Combinatorics and topology of proper toric maps

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Abstract. We study the topology of toric maps. We show that if $f: X \to Y$ is a proper toric morphism, with X simplicial, then the cohomology of every fiber of f is pure and of Hodge–Tate type. When the map is a fibration, we give an explicit formula for the Betti numbers of the fibers in terms of a relative version of the f-vector, extending the usual formula for the Betti numbers of a simplicial complete toric variety. We then describe the Decomposition Theorem for a toric fibration, giving in particular a nonnegative combinatorial invariant attached to each cone in the fan of Y, which is positive precisely when the corresponding closed subset of Y appears as a support in the Decomposition Theorem. The description of this invariant involves the stalks of the intersection cohomology complexes on X and Y, but in the case when both X and Y are simplicial, there is a simple formula in terms of the relative f-vector.

1. Introduction

A complex toric variety is a normal complex algebraic variety X that carries an action of a torus $T = (\mathbf{C}^*)^n$ such that X has an open orbit isomorphic to T. Toric varieties can be described in terms of convex-geometric objects, namely fans and polytopes, and part of their appeal comes from the fact that algebro-geometric properties of the varieties translate into combinatorial properties of the fans or polytopes, see [12]. For example, given a complete simplicial toric variety X (recall that *simplicial* translates as having quotient singularities), the information given by the Betti numbers of X is equivalent to that encoded by the f-vector of X, which records the number of cones of each dimension in the fan defining X. This has famously been used by Stanley in [20], together with the Poincaré duality and Hard Lefschetz theorems to prove a conjecture of McMullen concerning the f-vector of a simple polytope. When X is not-necessarily-simplicial, it turns out that the right cohomological invariant to consider is not the cohomology of X (which is *not* a combinatorial invariant), but the intersection cohomology (see [21] and also [10] and [11]).

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In this paper we are concerned with a relative version of this story. More precisely, we consider a proper (equivariant) morphism of toric varieties $f: X \to Y$ and study two related questions. We first study the cohomology of the fibers of f and then apply this study to describe the Decomposition Theorem for f. Our results are most precise when f is a fibration, that is, when it is proper, surjective and with connected fibers.

We begin with the following result about simplicial toric varieties that admit a proper toric map to an affine toric variety that has a fixed point. The case of complete simplicial toric varieties is well known. Recall that a pure Hodge structure of weight q is *of Hodge–Tate type* if all Hodge numbers $h^{i,j}$ are 0, unless $i = j$ (in particular, the underlying vector space is 0 if q is odd).

Theorem A.   *If X is a simplicial toric variety that admits a proper toric map to an affine toric variety that has a fixed point, then the following hold for every q:*

(i) *The canonical map $A_q(X)_{\mathbb{Q}} \to H^{\mathrm{BM}}_{2q}(X)_{\mathbb{Q}}$ is an isomorphism, where $A_q(X)_{\mathbb{Q}}$ is the Chow group of q-dimensional cycle classes and $H^{\mathrm{BM}}_{2q}(X)_{\mathbb{Q}}$ is the (2q)-th Borel–Moore homology group of X (both with $\mathbb{Q}$-coefficients).*

(ii) *The mixed Hodge structures on each of $H^q_c(X, \mathbb{Q})$ and $H^q(X, \mathbb{Q})$ are pure, of weight q, and of Hodge–Tate type.*

As a consequence, we deduce the following:

Theorem B.   *Let $f: X \to Y$ be a proper toric map between complex toric varieties, with X simplicial. For every $y \in Y$ and every q, the mixed Hodge structure on $H^q(f^{-1}(y), \mathbf{Q})$ is pure, of weight q, and of Hodge–Tate type.*

By virtue of Proposition 2.7 below, under the assumptions of Theorem B, every irreducible component of $f^{-1}(y)$ is a complete, simplicial toric variety, for which the properties in the statement are well known. However, the fact that the union still satisfies these properties is not a general fact. In order to prove Theorem B, we first reduce to the case when X is smooth, f is a projective fibration, Y is affine, and has a torus-fixed point which is equal to y. In this case, there is an isomorphism of mixed Hodge structures

$$H^q(f^{-1}(y), \mathbb{Q}) \simeq H^q(X, \mathbb{Q})$$

and therefore Theorem A implies Theorem B. In order to prove Theorem A, we prove Corollary 3.2, which yields a filtration of X by open subsets $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_r = X$ such that each difference $U_i \smallsetminus U_{i-1}$ is isomorphic to a quotient of an affine space by a finite group. The existence of such a filtration is well known when X is smooth and projective (see [12, Section 5.2]) and a similar augment gives it in the context we need. It is then easy to deduce the assertions about the cohomology of X from the existence of such a filtration. We remark that the purity statements in both theorems can also be deduced from the above isomorphism of mixed Hodge structures via some general properties of the weight filtration (see Remark 4.9). However, we have preferred to give the argument described above in order to emphasize the elementary nature of the results.

For every toric fibration $f: X \to Y$, it is easy to compute the Hodge–Deligne polynomial of the fibers of f. This is done in terms of a relative version of the familiar notion of f-vector

of a toric variety. Recall that the orbits of the torus action on a toric variety Y are in bijection with the cones in the fan Δ_Y defining Y, with the orbit $O(\tau)$ corresponding to $\tau \in \Delta_Y$ having dimension equal to $\mathrm{codim}(\tau)$. Moreover, the irreducible torus-invariant closed subsets of Y are precisely the orbit closures $V(\tau) = \overline{O(\tau)}$. With this notation, for every cone $\tau \in \Delta_Y$, we denote by $d_\ell(X/\tau)$ the number of irreducible torus-invariant closed subsets $V(\sigma)$ of X such that $f(V(\sigma)) = V(\tau)$ and $\dim(V(\sigma)) - \dim(V(\tau)) = \ell$. The following corollary (Corollary 4.7) generalizes the well-known formula for the Betti numbers of a complete, simplicial toric variety. Note that Theorem B implies that in what follows, the odd cohomology groups are trivial.

Corollary C. *If $f: X \to Y$ is a toric fibration, with X simplicial, and $y \in O(\tau)$ for some cone $\tau \in \Delta_Y$, then for every $m \in \mathbf{Z}_{\geq 0}$, we have*

$$\dim_{\mathbf{Q}} H^{2m}(f^{-1}(y), \mathbf{Q}) = \sum_{\ell \geq m} (-1)^{\ell-m} \binom{\ell}{m} d_\ell(X/\tau).$$

Moreover, the Euler–Poincaré characteristic of $f^{-1}(y)$ is given by $\chi(f^{-1}(y)) = d_0(X/\tau)$.

Our initial goal was to understand the decomposition theorem of [1] for proper toric morphisms. In the setting of finite fields, this is carried out in [7]. In our setting, i.e. over the complex numbers, the theorem takes the following form (for a variety X, we denote by IC_X the intersection complex on X; for nonsingular X, this is $\mathbf{Q}_X[\dim X]$).

Theorem D. *If X and Y are complex toric varieties and $f: X \to Y$ is a toric fibration, then we have a decomposition*

$$(1.1) \qquad Rf_*(\mathrm{IC}_X) \simeq \bigoplus_{\tau \in \Delta_Y} \bigoplus_{b \in \mathbb{Z}} \mathrm{IC}_{V(\tau)}^{\oplus s_{\tau,b}}[-b],$$

where the nonnegative integers $s_{\tau,b}$ satisfy $s_{\tau,b} = 0$ if $b + \dim(X) - \dim(V(\tau))$ is odd.

In fact, the integers $s_{\tau,b}$ in the above theorem satisfy further constraints coming from Poincaré duality and the relative Hard Lefschetz theorem; see Theorem 5.1 below for the precise statement. By comparison with the general statement of the decomposition theorem from [1], there are two points. Firstly, the subvarieties that appear in the decomposition (1.1) are torus-invariant and, secondly, the intersection complexes that appear have constant coefficients (this is due to the fact that the map is a fibration).

We are interested in the *supports* of toric fibrations, that is, the subvarieties $V(\tau)$ that appear in the decomposition (1.1). More precisely, in the setting of Theorem D, for every $\tau \in \Delta_Y$ we define the key invariant of this paper $\delta_\tau := \sum_{b \in \mathbb{Z}} s_{\tau,b}$. Clearly, in view of (1.1), we have that the closure of an orbit $V(\tau)$ is a support if and only if $\delta_\tau > 0$. Theorems E and F below relate the topological invariant δ of the toric fibration f to the associated combinatorial data.

Let us start by discussing the simpler case of Theorem E, where both varieties are simplicial. By building on Corollary C, we obtain the following description.

Theorem E. *If $f: X \to Y$ is a toric fibration, where X and Y are simplicial toric varieties, then for every cone $\tau \in \Delta_Y$, we have*

$$(1.2) \qquad \delta_\tau = \sum_{\sigma \subseteq \tau} (-1)^{\dim(\tau) - \dim(\sigma)} d_0(X/\sigma).$$

In fact, in this simplicial case, we obtain an explicit formula (see Theorem 6.1) for each of the numbers $s_{\tau,b}$. An interesting consequence of Theorem E is that the expression on the right-hand side of (1.2) is nonnegative. It would be desirable to find a direct combinatorial argument for this fact. When f is birational and $\dim(\tau) \leq 3$, we give a combinatorial description of δ_τ which implies that it is nonnegative (see Remark 6.5). However, we do not know of such a formula when τ has higher dimension.

It is worth noting that the explicit formula for the invariants $s_{\tau,b}$ in the simplicial case, in combination with Poincaré duality and Hard Lefschetz, leads to interesting constraints on the combinatorics of the morphism (see Remark 6.4 for the precise statement). These constraints extend to the relative setting the well-known conditions on the f-vector of a simplicial, projective toric variety.

In order to treat the case of fibrations between not-necessarily-simplicial toric varieties, we need to introduce some notation. Given a toric variety Y and two cones $\tau \subseteq \sigma$ in the fan Δ_Y defining Y, we put $r_{\tau,\sigma} := \dim_{\mathbb{Q}} \mathcal{H}^*(\mathrm{IC}_{V(\tau)})_{x_\sigma}$, where x_σ can be taken to be any point in the orbit $O(\sigma) \subseteq V(\tau)$. It is a consequence of the results in [11] and [10], independently, that $r_{\tau,\sigma}$ is a combinatorial invariant. In turn, we have the invariant $\widetilde{r}_{\tau,\sigma}$ for cones $\tau \subseteq \sigma$ in Δ_Y, uniquely determined by the property that for every $\tau \subseteq \sigma$, the sum $\sum_{\tau \subseteq \gamma \subseteq \sigma} r_{\tau,\gamma} \cdot \widetilde{r}_{\gamma,\sigma}$ is equal to 1 if $\tau = \sigma$ and it is equal to 0, otherwise. By [23], $\widetilde{r}_{\tau,\sigma}$ coincides, up to sign, with the $r_{\tau,\sigma}$-function of the dual poset. Suppose now that $f : X \to Y$ is a toric fibration. For a cone $\sigma \in \Delta_Y$, we put

$$p_{f,\sigma} := \dim_{\mathbb{Q}} \mathcal{H}^*(f^{-1}(x_\sigma), \mathrm{IC}_X),$$

where again we may take x_σ to be any point in the orbit $O(\sigma)$. The next result gives a description of δ_σ in terms of the above invariants. The second part implies that the invariants $p_{f,\sigma}$, hence also the δ_σ, are combinatorial.

Theorem F. *With the above notation, if $f : X \to Y$ is a toric fibration, then the following hold:*

(i) *For every cone $\tau \in \Delta_Y$, we have $\delta_\tau = \sum_{\sigma \subseteq \tau} \widetilde{r}_{\sigma,\tau} \cdot p_{f,\sigma}$.*

(ii) *For every cone $\sigma \in \Delta_Y$, we have $p_{f,\sigma} = \sum_i r_{0,\sigma_i}$, where the σ_i are the cones in the fan Δ_X defining X with the property that $f(V(\sigma_i)) = V(\sigma)$ and $\dim(V(\sigma_i)) = \dim(V(\sigma))$.*

Just as for Theorem E, an explicit formula for each of the numbers $s_{\tau,b}$ is obtained by upgrading $r_{\sigma,\tau}$ and $p_{f,\sigma}$ to Laurent polynomials with integral coefficients in order to keep track of the cohomological grading (see Theorem 7.3 for the precise result). Again, as in Theorem E, note that while $\delta \geq 0$ by definition, the right-hand side of Theorem F (i) contains the factors $\widetilde{r}$, which have "alternating signs", so that, in particular, we find this right-hand side to be nonnegative as well. In this paper, this is proved as a consequence of the decomposition theorem. As in the simplicial case, it would be desirable to find a combinatorial description for δ that implies its non-negativity.

We mention that in their recent preprint [15], Katz and Stapledon undertake a related study in a more combinatorial framework, focused on invariants associated to certain maps between posets. In the case of a toric fibration $f : X \to Y$, their results apply to the map $f_* : \Delta_X \to \Delta_Y$. In particular, our invariants $s_{\tau,b}$ appear in their setting as the coefficients of a *local h-polynomial*. For some comparisons between their results and ours, see Remarks 5.7, 6.2, and 7.7.

The paper is organized as follows. In Section 2 we review the basics of toric geometry that we use. We pay special attention to some facts concerning toric morphisms that seem somewhat less known, such as the description of toric fibrations and Stein factorizations in the toric setting, and the description of the irreducible components of fibers of toric maps. In Section 3 we study the cohomology of toric varieties that admit proper toric maps to affine toric varieties that have a fixed point. In particular, we prove Theorem A (see Theorems 3.5 and 4.1). In Section 4, we apply the results in the previous section to obtain Theorem B (see Theorem 4.1). We use this and the computation of Hodge–Deligne polynomials for fibers of toric fibrations to give the formula for the Betti numbers of such fibers in Corollary C (see Corollary 4.7). In Section 5 we deduce Theorem D from the decomposition theorem in [1] (see Theorem 5.1). Section 6 is devoted to the description of the invariants δ_σ in the case of a toric fibration between simplicial toric varieties (see Theorem 6.1), while in Section 7 we treat the general case (see Theorems 7.3 and 7.6).

2. Basic toric algebraic geometry

In this section we review some basic facts about toric varieties and toric maps. For all assertions that we do not prove in this section, as well as for all the standard notation for toric varieties that we employ, we refer to [12]. We work over an algebraically closed field k, of arbitrary characteristic, in the hope that some of the facts that we prove might be useful somewhere else, in this more general setting.

Toric varieties and their orbits.   A toric variety X is associated with a lattice N and a fan Δ in $N_\mathbb{R} = N \otimes_\mathbf{Z} \mathbb{R}$. If M is the dual lattice of N, then the torus $T_N := \operatorname{Spec} k[M]$ embeds as an open subset in X and its standard action on itself extends to an action on X. We often write N_X, M_X, Δ_X, and T_X for the objects corresponding to a fixed toric variety X. For every cone $\sigma \in \Delta$, there is an affine open subset U_σ of X, with $U_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$. These affine open subsets cover X. The support of a fan Δ is the subset $|\Delta| = \bigcup_{\sigma \in \Delta} \sigma$ of $N_\mathbb{R}$. The toric variety X is complete if and only if $|\Delta_X| = N_\mathbb{R}$.

The orbits of the T_X-action on X are in bijection with the cones in Δ_X. The orbit $O(\sigma) := \operatorname{Spec} k[M_X \cap \sigma^\perp]$ corresponding to $\sigma \in \Delta_X$ is a torus of dimension equal to $\operatorname{codim}(\sigma)$. The distinguished element of $O(\sigma)$ (the identity of the group) is denoted by x_σ. In particular, the smallest cone $\{0\}$ corresponds to the open orbit T_X.

The irreducible torus-invariant closed subsets of X are precisely the orbit closures $V(\sigma) := \overline{O(\sigma)}$. The lattice corresponding to $V(\sigma)$ is N/N_σ, where N_σ is the intersection of N with the linear span of σ. We always view Δ as a poset, ordered by the inclusion of cones.

Note that $\sigma \subseteq \tau$ if and only if $V(\sigma) \supseteq V(\tau)$. The open subset U_σ is the union of those $O(\tau)$ with $\tau \subseteq \sigma$.

Each $V(\sigma)$ is a toric variety, with corresponding torus $O(\sigma)$. There is a surjective morphism of algebraic groups $T_X \to O(\sigma)$ such that the T_X-action on $V(\sigma)$ induces the $O(\sigma)$-action. This morphism corresponds to the split inclusion $M_X \cap \sigma^\perp \hookrightarrow M_X$. If X is smooth or simplicial, then each $V(\sigma)$ has the same property.

We say that an affine toric variety U_σ is *of contractible type* if σ is a cone that spans $N_\mathbb{R}$. Note that in this case $x_\sigma \in U_\sigma$ is the unique fixed point for the torus action.

Toric maps. Let X and Y be toric varieties corresponding, respectively, to the lattices N_X and N_Y and to the fans Δ_X and Δ_Y. A toric map is a morphism $f : X \to Y$ that induces a morphism of algebraic groups $g : T_X \to T_Y$ such that f is T_X-equivariant with respect to the T_X-action on Y induced by g. Such f corresponds to a unique linear map $f_{N_\mathbb{R}} : (N_X)_\mathbb{R} \to (N_Y)_\mathbb{R}$ inducing $f_N : N_X \to N_Y$ such that for every cone $\sigma \in \Delta_X$, there is a cone $\tau \in \Delta_Y$ with $f_{N_\mathbb{R}}(\sigma) \subseteq \tau$. We write $f_M : M_Y \to M_X$ for the dual of f_N. Note that for σ and τ as above, we have a k-algebra homomorphism $k[M_Y \cap \tau^\vee] \to k[M_X \cap \sigma^\vee]$ mapping χ^u to $\chi^{f_M(u)}$. This induces a morphism $U_\sigma \to U_\tau$, which is the restriction of f to U_σ. In general, we have $f_{N_\mathbb{R}}^{-1}(|\Delta_Y|) \supseteq |\Delta_X|$ and the map f is proper if and only if $f_{N_\mathbb{R}}^{-1}(|\Delta_Y|) = |\Delta_X|$. Note that by definition, we have a map $f_* : \Delta_X \to \Delta_Y$ such that $f_*(\sigma)$ is the smallest cone in Δ_Y that contains $f_N(\sigma)$. It is clear that f_* is a map of posets.

Suppose now that $f : X \to Y$ is a toric map such that the lattice map $f_N : N_X \to N_Y$ is surjective. In this case the morphism of tori $T_X \to T_Y$ is surjective. More generally, if σ is a cone in Δ_X, then f induces a surjective morphism of tori $O(\sigma) \to O(\tau)$, where $\tau = f_*(\sigma)$. We denote the kernel of this morphism by $O(\sigma/\tau)$. Note that this is again a torus, of dimension $\operatorname{codim}(\sigma) - \operatorname{codim}(\tau)$. It is clear that $f(V(\sigma)) \subseteq V(\tau)$ (with equality if f is proper). In fact, the induced map $V(\sigma) \to V(\tau)$ is again a toric map of toric varieties with the property that the corresponding lattice map is surjective.

Toric Stein factorizations. Recall that a proper morphism $f : X \to Y$ is a *fibration* if $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. This implies that f is surjective and has connected fibers; the converse holds if $\operatorname{char}(k) = 0$. The Stein factorization of a proper map $f : X \to Y$ is the unique factorization as

$$X \xrightarrow{g} Z \xrightarrow{h} Y$$

such that g is a fibration and h is a finite map. The following proposition gives the description of fibrations and Stein factorizations in the toric setting.

Proposition 2.1. *Let $f : X \to Y$ be a proper toric map corresponding to the lattice map $f_N : N_X \to N_Y$ and let Δ_X and Δ_Y be the corresponding fans.*

(i) *The map f is surjective if and only if $\operatorname{Coker}(f_N)$ is finite.*

(ii) *The map f is a fibration if and only if f_N is surjective.*

(iii) *Suppose that f is surjective. If we let $N_Z = f_N(N_X)$ and $\Delta_Z = \Delta_Y$, then the factorization*

$$N_X \xrightarrow{g_N} N_Z \xrightarrow{h_N} N_Y$$

of f_N induces the Stein factorization of f.

Proof. Since f is proper, it is surjective if and only if the induced morphism of tori $T_X \to T_Y$ is dominant. This is the case if and only if $f_M \colon M_Y \to M_X$ is injective, which is equivalent to $\mathrm{Coker}(f_N)$ being finite. This proves (i).

Suppose now that f_N is surjective. In order to show that f is a fibration, it is enough to prove that for every $\sigma \in \Delta_Y$, the natural map

$$\Gamma(U_\sigma, \mathcal{O}_Y) = k[\sigma^\vee \cap M_Y] \to \Gamma(f^{-1}(U_\sigma), \mathcal{O}_X)$$

is an isomorphism. Note that $f^{-1}(U_\sigma)$ is the union of U_τ, where τ varies over the set Λ_σ of those cones in Δ_X such that $f_N(\tau) \subseteq \sigma$. Therefore

$$\Gamma(f^{-1}(U_\sigma), \mathcal{O}_X) = \bigcap_{\tau \in \Lambda_\sigma} k[\tau^\vee \cap M_X] = \bigoplus_u k \cdot \chi^u,$$

where the direct sum is over those $u \in M_X$ such that $u \in \tau^\vee$ for every $\tau \in \Lambda_\sigma$. It is enough to show that for every such $u \in M_X$, there is $w \in M_Y \cap \sigma^\vee$ such that $f_M(w) = u$ (note that w is clearly unique since f_M is injective).

It is clear that there is $w \in M_Y$ such that $f_M(w) = u$. Indeed, since $f_{N_\mathbb{R}}^{-1}(|\Delta_Y|) = |\Delta_X|$, we deduce that $\mathrm{Ker}(f_{N_\mathbb{R}})$ is a union of cones in Δ_X, which automatically lie in Λ_σ. Therefore $u \in (\mathrm{Ker}(f_N))^\perp = \mathrm{Im}(f_M)$. We need to show that w lies in $\sigma^\vee$. Let $v \in \sigma \cap N_Y$. Since f_N is surjective, we can write $v = f_N(\widetilde{v})$ for some $\widetilde{v} \in N_X$. Since $\widetilde{v} \in f_{N_\mathbb{R}}^{-1}(|\Delta_Y|) = |\Delta_X|$, we may choose $\tau \in \Delta_X$ smallest such that $\widetilde{v} \in \tau$. In this case $f_{N_\mathbb{R}}(\tau) \subseteq \sigma$. Therefore $u \in \tau^\vee$, which implies

$$0 \le \langle u, \widetilde{v} \rangle = \langle f_M(w), \widetilde{v} \rangle = \langle w, f_N(\widetilde{v}) \rangle = \langle w, v \rangle$$

and we see that indeed $w \in \sigma^\vee$.

Suppose now that f is surjective and consider the decomposition $f = h \circ g$ in (iii). It is straightforward to see that h is finite, while we have already shown that g is a fibration. Therefore this decomposition is the Stein factorization of f. We also deduce from the uniqueness of the Stein factorization that if f is a fibration, then f_N is surjective. This completes the proof of the proposition.   □

Remark 2.2.   A finite, surjective toric morphism $h \colon Z \to Y$ can be described as follows. Note that h_N is injective, with finite cokernel. Suppose first that $\mathrm{char}(k)$ does not divide the order of $|\mathrm{Coker}(h_N)|$, which is equal to the order of $A := M_Z/M_Y$. In this case, h is the quotient by a faithful action of $G = \mathrm{Spec}\, k[A]$. In particular, h is generically a Galois cover.

Indeed, note first that the assumption on the characteristic of k implies that G is a reduced scheme, isomorphic to the finite group $\mathrm{Hom}(A, k^*)$. We claim that G acts faithfully on Z such that Y is the quotient by this group action. It is enough to check this on affine open subsets, hence we may assume that $Y = U_\sigma$. The morphism h corresponds to

$$k[\sigma^\vee \cap M_Y] \hookrightarrow k[\sigma^\vee \cap M_Z].$$

The G-action on Z corresponds to

$$k[\sigma^\vee \cap M_Z] \to k[M_Z/M_Y] \otimes_k k[\sigma^\vee \cap M_Z], \quad \chi^u \to \chi^{\bar{u}} \otimes \chi^u,$$

where $\bar{u}$ is the class of u in M_Z/M_Y. It is clear from this definition that the action is faithful and furthermore, that

$$k[\sigma^\vee \cap M_Z]^G = k[\sigma^\vee \cap M_Y],$$

as claimed.

When $\mathrm{char}(k) = p > 0$, given an arbitrary surjective, finite toric morphism $h\colon Z \to Y$, we can uniquely factor $h_M\colon M_Y \hookrightarrow M_Z$ as

$$M_Y \hookrightarrow M_{\widetilde{Z}} \hookrightarrow M_Z$$

such that the order of $M_Z/M_{\widetilde{Z}}$ is relatively prime to p and $M_{\widetilde{Z}}/M_Y$ is a product of abelian p-groups. This corresponds to a factorization of f as

$$Z \longrightarrow \widetilde{Z} \xrightarrow{\ \alpha\ } Y,$$

with $Z \to \widetilde{Z}$ a quotient as described above and α a universal homeomorphism. In fact, there is $\beta\colon Y \to \widetilde{Z}$ and $m \geq 1$ such that

$$\beta \circ \alpha = \mathrm{Frob}_{\widetilde{Z}}^m \quad \text{and} \quad \alpha \circ \beta = \mathrm{Frob}_Y^m$$

(note that a toric variety W is defined over the prime field, hence it is endowed with a Frobenius morphism Frob_W that is linear over the ground field). In order to see this, let us choose a basis $v_1, \ldots, v_n$ for N_Y such that $p^{m_1} v_1, \ldots, p^{m_n} v_n$ is a basis for $N_{\widetilde{Z}}$, for some positive integers $m_1, \ldots, m_n$. If $m = \max_i m_i$ and Z' is the toric variety corresponding to the lattice spanned by $p^{m_1 - m} e_1, \ldots, p^{m_n - m} e_n$ (the fan being the same as that of the toric varieties Z, $\widetilde{Z}$, and Y), then multiplication by p^m induces an isomorphism of toric varieties $Z' \simeq \widetilde{Z}$. If $\beta\colon Y \to Z' \simeq \widetilde{Z}$ is the induced morphism, then it is easy to check that it has the desired properties.

Remark 2.3. Suppose that $f\colon X \to Y$ is any proper toric map, possibly not surjective. In this case f has a canonical factorization

$$X \xrightarrow{\ u\ } W \xrightarrow{\ w\ } Y$$

such that both u and w are proper toric maps, with u surjective and w finite, with $T_W \to T_Y$ a closed immersion. Indeed, if $w\colon W \to Y$ is the normalization of $f(X)$, then since X is normal, there is a unique morphism $u\colon X \to W$ such that $f = w \circ u$. If

$$N_W := \{v \in N_Y : mv \in f_N(N_X) \text{ for some } m \in \mathbf{Z}_{>0}\}$$

and T is the torus corresponding to N_W, then the restriction of f to T_X factors as

$$T_X \xrightarrow{\ \phi\ } T \xrightarrow{\ \psi\ } T_Y,$$

with ϕ surjective and ψ a closed immersion. Therefore T is equal to $f(T_X)$. It is easy to deduce from Chevalley's constructibility theorem that T is an open dense subset of $f(X)$, hence T admits an open immersion in W. Moreover, the map $T \times f(X) \to f(X)$ given by the T_X-action on Y induces a map $T \times W \to W$ giving an action of T on W that extends the standard action of T on itself. Since by construction W is separated and normal, it follows that W is a toric variety with torus T. Moreover, both u and w are toric maps.

Remark 2.4. Let us consider the above decompositions of toric maps in the case of a morphism of algebraic groups $f\colon T_1 \to T_2$ between tori. We have a decomposition

$$T_1 \xrightarrow{\ \phi_1\ } A \xrightarrow{\ \phi_2\ } B \xrightarrow{\ \phi_3\ } C \xrightarrow{\ \phi_4\ } T_2$$

such that the following hold:

(i) There is an isomorphism $T_1 \simeq A \times A'$, with A' a torus, such that ϕ_1 corresponds to the projection onto the first component.

(ii) ϕ_2 is finite, surjective, and étale, the quotient by the action of a finite group.

(iii) If $\text{char}(k) = 0$, then ϕ_3 is an isomorphism. If $\text{char}(k) = p > 0$, then there is $\beta \colon C \to B$ and $m \geq 1$ such that $\beta \circ \phi_3 = \text{Frob}_B^m$ and $\phi_3 \circ \beta = \text{Frob}_C^m$.

(iv) ϕ_4 is a closed immersion.

Remark 2.5. We say that a toric variety X has *convex, full-dimensional fan support* if $|\Delta_X|$ is a convex cone in $(N_X)_{\mathbb{R}}$ (automatically rational polyhedral), of maximal dimension. Note that if X admits a proper morphism $f \colon X \to Y$, where Y is an affine toric variety of contractible type, then X has convex, full-dimensional fan support. Conversely, if this is the case, then we can find f as above. In fact, we may take f to be a fibration: if Λ is the largest linear subspace contained in $|\Delta_X|$ and $\sigma = |\Delta_X|/\Lambda$, considered as a cone with respect to the lattice $N_X/N_X \cap \Lambda$, then $Y = U_\sigma$ is of contractible type and we have a canonical toric fibration $f \colon X \to Y$. Finally, we note that given a proper morphism $f \colon X \to Y$, with Y affine, f is projective if and only if X is quasi-projective.

By Proposition 2.1 and Remark 2.3, we can reduce studying the fibers of an arbitrary proper toric map $f \colon X \to Y$ to the case of a fibration. Because of this, we will mostly consider this case.

Fibers of toric maps.   We want to show that the irreducible components of the fibers of a toric map are toric varieties. We begin by reviewing some basic facts that will be used also later, when reducing to the case of affine toric varieties of contractible type.

Recall that if X is a toric variety defined by the fan Δ in $N_{\mathbb{R}}$ and N' is a finitely generated subgroup of N such that N/N' is free and the linear span of $|\Delta|$ is contained in $N'_{\mathbb{R}}$, then we have a toric variety X' corresponding to Δ, considered as a fan in $N'_{\mathbb{R}}$. The inclusion $\iota \colon N' \hookrightarrow N$ induces a toric map $X' \to X$. In fact, if we choose a splitting of ι, we get an isomorphism $N \simeq N' \times N/N'$ and a corresponding isomorphism $T_N \simeq T_{N'} \times T_{N/N'}$. Moreover, we get an isomorphism $X \simeq X' \times T_{N/N'}$ compatible with the decomposition of T_N.

A special case that we will often use is that when $X = X(\Delta)$ is an arbitrary toric variety, σ is a cone in Δ and $N' = N_\sigma$. Applying the previous considerations to U_σ, we obtain an isomorphism of tori and an isomorphism of affine toric varieties

$$(2.1) \qquad T_N \simeq T_{N_\sigma} \times O(\sigma), \quad U_\sigma \simeq U_{\sigma'} \times O(\sigma),$$

where σ' is the same as σ, but considered in $N'_{\mathbb{R}}$.

We now turn to a version of such product decompositions in the relative setting.

Lemma 2.6. *Let $f \colon X \to Y$ be a toric fibration. Given $\tau \in \Delta_Y$, let us choose a splitting of $(N_Y)_\tau \hookrightarrow N_Y$ and then a splitting of f_N. These determine equivariant isomorphisms*

$$U_\tau \simeq U_{\tau'} \times O(\tau) \quad \text{and} \quad f^{-1}(U_\tau) \simeq f^{-1}(U_{\tau'}) \times O(\tau)$$

such that $f^{-1}(U_\tau) \to U_\tau$ gets identified to $f_{\tau'} \times \text{Id}$, where $f_{\tau'} \colon f^{-1}(U_{\tau'}) \to U_{\tau'}$ is the restric-

tion of f over $U_{\tau'}$. Moreover, the following hold:

(i) *We have an induced isomorphism of tori $T_{U_\tau} \simeq T_{U_{\tau'}} \times O(\tau)$.*

(ii) *We have an isomorphism $f^{-1}(O(\tau)) \simeq f^{-1}(x_\tau) \times O(\tau)$ such that the restriction of f to $f^{-1}(O(\tau))$ corresponds to the projection onto the second component. In particular, $f^{-1}(y) \simeq f^{-1}(x_\tau)$ for every $y \in O(\tau)$.*

(iii) *The map $f_{\tau'}$ is a toric fibration over an affine toric variety of contractible type and we have an isomorphism*

$$f^{-1}(x_\tau) \simeq f_{\tau'}^{-1}(x_{\tau'}).$$

Proof. Let $N_Y' = (N_Y)_\tau$ and $N_X' = f_N^{-1}(N_Y')$. Note that $f^{-1}(U_\tau) = \bigcup_\sigma U_\sigma$, where the union is over those $\sigma \in \Delta_X$ such that $f_*(\sigma) \subseteq \tau$. Therefore the linear span of the support of the fan defining $f^{-1}(U_\sigma)$ is contained in $(N_X')_{\mathbb{R}}$.

The two splittings in the lemma induce isomorphisms

$$N_Y \simeq N_Y' \times N_Y/N_Y' \quad \text{and} \quad N_X \simeq N_X' \times N_Y/N_Y'$$

such that f_N corresponds to $f_N' \times \mathrm{Id}$, where $f_N' \colon N_X' \to N_Y'$ is the induced lattice map. By applying (2.1) to both U_σ and $f^{-1}(U_\sigma)$ with respect to the subgroups N_Y' and N_X', respectively, we obtain the assertions in the lemma. □

The following proposition describes the irreducible components for the fibers of proper toric maps.

Proposition 2.7. *Let $f \colon X \to Y$ be a proper toric map.*

(i) *Every irreducible component of a fiber $f^{-1}(y)$ is a toric variety. Moreover, this is smooth or simplicial if X has this property.*

(ii) *If f is a fibration and $y \in O(\tau)$ for some $\tau \in \Delta_Y$, then $f^{-1}(y)$ is a disjoint union of locally closed subsets parametrized by the cones $\sigma \in \Delta_X$ such that $f_*(\sigma) = \tau$, with the subset corresponding to σ being isomorphic to the torus $O(\sigma/\tau)$.*

Proof. It follows from Proposition 2.1 and Remark 2.3 that we may write f as a composition

$$X \xrightarrow{g} Z \xrightarrow{h} Y,$$

with g a fibration and h finite. Since a fiber of f is either empty or a disjoint union of fibers of g, it follows that in order to prove (i), we may and will assume that f is a fibration.

Let $\tau \in \Delta_Y$ and suppose that $y \in O(\tau)$. It follows from Lemma 2.6 that

$$f^{-1}(y) \simeq f^{-1}(x_\tau),$$

hence we may assume that $y = x_\tau$. Moreover, we have an isomorphism

$$(2.2) \qquad f^{-1}(O(\tau)) \simeq f^{-1}(x_\tau) \times O(\tau).$$

It follows that if W is an irreducible component of $f^{-1}(x_\tau)$, then we get an induced isomorphism $V \simeq W \times O(\tau)$, where V is an irreducible component of $f^{-1}(O(\tau))$. Note that $f^{-1}(O(\tau))$ is preserved by the T_X-action, hence V has the same property since T_X is connected. We conclude that the closure $\overline{V}$ is equal to $V(\gamma)$ for some cone $\gamma \in \Delta_X$. Note that V is locally closed in X, hence it is open in $\overline{V}$; therefore it is a toric variety with torus $O(\gamma)$.

Since $O(\gamma) \subseteq f^{-1}(O(\tau))$, we see that $f_*(\gamma) = \tau$. Moreover, the isomorphism (2.2) induces an isomorphism $O(\gamma) \simeq O(\gamma/\tau) \times O(\tau)$ and it is now clear that W is a toric variety with torus $O(\gamma/\tau)$. Note that if X is smooth or simplicial, then $V(\gamma)$ has the same property and so does V. In this case W is smooth, respectively simplicial, as well. This completes the proof of (i).

In order to check the assertion in (ii), note that $f^{-1}(O(\tau))$ is the union of the orbits $O(\sigma)$, where σ runs over the cones of Δ_X such that $f_*(\sigma) = \tau$. For every such σ, the isomorphism (2.2) induces an isomorphism $O(\sigma) \simeq Z_\sigma \times O(\tau)$ such that Z_σ is isomorphic to $O(\sigma/\tau)$. Since $f^{-1}(x_\tau)$ is the disjoint union of the locally closed subsets Z_σ, this completes the proof of the proposition. $\square$

The toric Chow lemma.   We will make use of the following toric version of Chow's lemma. For a proof in the case when X is complete, see [5, Theorem 6.1.18]. For the general case, see [24, Theorem 2].

Proposition 2.8.   *If X is a toric variety, then there is a projective toric birational morphism $f: \widetilde{X} \to X$ such that $\widetilde{X}$ is quasi-projective.*

Remark 2.9.   By combining the toric Chow lemma with toric resolution of singularities, we deduce that for every toric variety X, there is a projective birational morphism $\pi: \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and quasi-projective.

3. Cohomology of simplicial toric varieties with convex, full-dimensional fan support

Our goal in this section is to show that if X is a complex simplicial toric variety such that the support $|\Delta_X|$ is a full-dimensional, convex cone, then the cohomology of X behaves similarly to the case when X is complete (we refer to [12, Section 5.2] for that case). Recall that by Remark 2.5, saying that $|\Delta_X|$ is convex and full-dimensional is equivalent to saying that X admits a proper morphism to an affine toric variety, of contractible type.

A good filtration by open subsets.   The key ingredient in this study is a certain filtration of X by open subsets, in the case when X is quasi-projective. This filtration has been well understood and used in the case when X is projective (see for example [16] and [11]). We begin by showing that such a filtration also exists when X is quasi-projective and has convex, full-dimensional fan support. While we work over $\mathbf{C}$, we remark that the results of this subsection hold over any algebraically closed field.

Let X be a complex quasi-projective toric variety, with convex, full-dimensional fan support. We fix an ample torus-invariant Cartier divisor D on X and consider the corresponding polyhedron $P = P_D$ (note that P might not be bounded since X might not be complete).

We first recall some basic facts related to this setting. Given D, to each maximal cone $\sigma \in \Delta_X$ one associates an element $u(\sigma)$ in the lattice M_X such that $D|_{U_\sigma} = \operatorname{div}(\chi^{-u(\sigma)})$. If P_0 is the convex hull of the $u(\sigma)$, where σ varies over the maximal cones of Δ_X, then $P = P_0 + |\Delta_X|^\vee$. If Q is a face of P, then we get a cone $\sigma_Q \in \Delta_X$ defined by

$$\sigma_Q = \{w \in (N_X)_\mathbb{R} : \langle u, w \rangle \le \langle u', w \rangle \text{ for all } u \in Q, \, u' \in P\}.$$

Moreover, each cone in Δ_X corresponds in this way to a unique face of P. Note that for every maximal cone $\sigma \in \Delta_X$, the corresponding $u(\sigma)$ is a vertex of P and the cone corresponding to $u(\sigma)$ is σ, that is, $\sigma^\vee$ is generated by $\{u - u(\sigma) : u \in P\}$. All these facts are well known when X is complete. The proofs easily extend to our setting, see [17, Section 6].

Suppose now that $v \in |\Delta_X| \cap N_X$ is such that the following conditions are satisfied:

(C1) v is not orthogonal to any minimal generator of the pointed cone $|\Delta_X|^\vee$ (equivalently, v does not lie on any facet of $|\Delta_X|$).

(C2) The integers $\langle u(\sigma), v \rangle$, when σ varies over the maximal cones in Δ_X, are mutually distinct.

We denote by $\gamma_v \colon \mathbb{C}^* \to T_X$ the one-parameter subgroup of T_X corresponding to v. Suppose that $\sigma_1, \ldots, \sigma_r$ are the maximal cones in Δ, ordered such that

$$(3.1) \qquad \langle u(\sigma_1), v \rangle < \cdots < \langle u(\sigma_r), v \rangle$$

(condition (C2)) above implies that there is such an ordering and this is of course unique). For every i, with $1 \le i \le r$, we denote by x_i the torus-fixed point in U_{σ_i}.

Proposition 3.1.   *With the above notation, the following hold:*

(i) *The only fixed points for the $\mathbb{C}^*$-action on X induced by γ_v are $x_1, \ldots, x_r$.*

(ii) *For every i, let X_i consist of those $x \in X$ such that the map $\gamma_{v,x} \colon \mathbb{C}^* \to X$, given by*

$$\gamma_{v,x}(t) = \gamma_v(t) \cdot x,$$

extends to a map $\widetilde{\gamma}_{v,x} \colon \mathbf{A}^1 \to X$ with $\widetilde{\gamma}_{v,x}(0) = x_i$. If $U_i := \bigcup_{j \le i} X_j$, then U_i is open in X for every i and $U_r = X$.

Proof.   It is clear that $x_1, \ldots, x_r$ are fixed points for the $\mathbb{C}^*$-action since they are fixed by the T_X-action. Therefore in order to prove (i) it is enough to show that if $x \in U_{\sigma_i}$ is a $\mathbb{C}^*$-fixed point, then $x = x_i$. By definition, the action of γ_v on U_{σ_i} is such that

$$(3.2) \qquad \chi^u(\gamma_v(t) \cdot x) = t^{\langle u, v \rangle} \chi^u(x)$$

for every $t \in \mathbb{C}^*$ and $u \in \sigma_i^\vee \cap M_X$. Since x is a fixed point for the $\mathbb{C}^*$-action, it follows that $\langle u, v \rangle = 0$ for all $u \in \sigma_i^\vee \cap M_X$ such that $\chi^u(x) \ne 0$.

We need to show that $S := \{u \in \sigma_i^\vee \cap M_X : \chi^u(x) \ne 0\}$ is equal to $\{0\}$. We have seen that $S \subseteq v^\perp$. Note that for $u_1, u_2 \in \sigma_i^\vee \cap M_X$, we have $u_1 + u_2 \in S$ if and only if $u_1, u_2 \in S$. Since $\sigma_i^\vee$ is generated as a convex cone by $|\Delta_X|^\vee$ and by $\{u(\sigma_j) - u(\sigma_i) : j \ne i\}$, it follows that if $S \ne \{0\}$, then either v is orthogonal to a ray of $|\Delta_X|^\vee$ or it is orthogonal to some $u(\sigma_j) - u(\sigma_i)$ with $j \ne i$. Since both these conclusions contradict conditions (C1) and (C2) above, we conclude that $S = \{0\}$, completing the proof of (i).

We note that for every $x \in X$, the map $\gamma_{v,x} \colon \mathbb{C}^* \to X$ extends to a map $\mathbf{A}^1 \to X$. Indeed, consider the canonical toric fibration $f \colon X \to Y$, where Y is an affine toric variety of contractible type (see Remark 2.5). Since Y is affine and the image of v in N_Y lies in the cone defining Y, the composition $f \circ \gamma_{v,x}$ extends to a map $\mathbf{A}^1 \to Y$ (see [12, Section 2.3]). Since f is proper, the valuative criterion for properness implies that $\gamma_{v,x}$ extends to a map $\widetilde{\gamma}_{v,x} \colon \mathbf{A}^1 \to X$. Since $\widetilde{\gamma}_{v,x}(0)$ is clearly fixed by the $\mathbb{C}^*$-action induced by γ_v, it follows from (i) that it is equal to one of the x_i. Therefore $\bigcup_{i=1}^r X_i = X$.

Let us describe now X_i. If $x \in X_i$, then it is clear that $x \in U_{\sigma_i}$. Suppose now that $x \in U_{\sigma_i}$ is arbitrary. Using again formula (3.2), we conclude that $\gamma_{v,x}(0) = x_i$ if and only if for every $u \in (\sigma_i^\vee \cap M_X) \smallsetminus \{0\}$ with $\chi^u(x) \neq 0$, we have $\langle u, v \rangle > 0$. Let τ be the face of σ_i such that $x \in O(\tau)$. In this case, for $u \in \sigma_i^\vee \cap M_X$ we have $\chi^u(x) \neq 0$ if and only if $u \in \sigma_i^\vee \cap \tau^\perp \cap M_X$. We thus conclude that $X_i = \bigcup_\tau O(\tau)$, where the union is over the faces τ of σ_i such that $\langle u, v \rangle > 0$ for every $u \in (\sigma_i^\vee \cap \tau^\perp \cap M_X) \smallsetminus \{0\}$.

On the other hand, a face τ of σ_i is of the form σ_Q for a unique face Q of P such that $u(\sigma_i) \in Q$. In this case $\sigma_i^\vee \cap \tau^\perp$ is generated by $\{u - u(\sigma_i) : u \in Q\}$. We deduce that X_i is the union of those $O(\sigma_Q)$, over the faces Q of P containing $u(\sigma_i)$ having the property that $\langle u, v \rangle > \langle u(\sigma_i), v \rangle$ for every $u \in Q$, with $u \neq u(\sigma_i)$. On the other hand, since $P = |\Delta_X|^\vee + P_0$, we see that for every face Q of P, an element $u \in Q$ can be written as $u = u' + \sum_i \lambda_i u(\sigma_i)$, with $u' \in |\Delta_X|^\vee$ and $\lambda_i \in \mathbb{R}_{\geq 0}$, with $\sum_i \lambda_i = 1$. Since $v \in |\Delta_X|$, it follows from the way we have ordered the maximal cones that

$$O(\sigma_Q) \subseteq X_i \quad \text{if and only if} \quad i = \min\{j : u(\sigma_j) \in Q\}.$$

We conclude that U_i is the union of those orbits $O(\sigma_Q)$ (with Q a face of P) such that some $u(\sigma_j)$, with $j \leq i$, lies in Q. It is clear that if Q has this property, then any face of P that contains Q also has this property. Equivalently, U_i is a union of orbits such that $O(\sigma) \subseteq U_i$, then $O(\tau) \subseteq U_i$ for every face τ of σ. This implies that U_i is open. Since

$$U_r = X_1 \cup \cdots \cup X_r = X,$$

this completes the proof of (ii). □

Corollary 3.2. *If X is a quasi-projective toric variety, with convex, full-dimensional fan support, then there are open torus-invariant subsets*

$$\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_r = X$$

such that for every $i \geq 1$, the set $U_i \smallsetminus U_{i-1}$ is a closed subset of some U_σ, where σ is a maximal cone in Δ_X. Moreover, if X is smooth (resp. simplicial), then each $U_i \smallsetminus U_{i-1}$ is an affine space (resp. the quotient of an affine space by a finite group).

Proof. We consider the U_i given by Proposition 3.1, hence $U_i \smallsetminus U_{i-1} = X_i$. We have seen in the proof of the proposition that $X_i \subseteq U_{\sigma_i}$ and it follows from the definition that

$$(3.3) \qquad X_i = \{x \in U_{\sigma_i} : \chi^u(x) = 0 \text{ if } u \in (\sigma_i^\vee \cap M_X) \smallsetminus \{0\}, \langle u, v \rangle \leq 0\}.$$

Therefore X_i is closed in U_{σ_i}.

Suppose now that X is simplicial and let $v_1, \ldots, v_n$ be the primitive generators of the rays in σ_i. If $w_1, \ldots, w_n \in M_X$ are such that $\langle w_j, v_j \rangle > 0$ for all j and $\langle w_j, v_k \rangle = 0$ for $j \neq k$, then it is easy to deduce from (3.3) that

$$X_i = \{x \in U_{\sigma_i} : \chi^{w_j}(x) = 0 \text{ if } \langle w_j, v \rangle \leq 0\}$$

(note that for every nonzero $u \in \sigma_i^\vee \cap M_X$, some positive multiple of u can be written as $\sum_j a_j w_j$, with $a_j \in \mathbb{Z}_{\geq 0}$, and if $\langle u, v \rangle \leq 0$, then there is j with $a_j > 0$ and $\langle w_j, v \rangle \leq 0$). This implies that $X_i = U_{\sigma_i} \cap V(\tau_i)$, where τ_i is the face of σ_i spanned by those v_j, with j such that $\langle w_j, v \rangle \leq 0$. In particular, X_i is a simplicial affine toric variety of contractible type (smooth, if X is smooth). This completes the proof of the corollary. □

Remark 3.3.   The hypothesis that X is quasi-projective is crucial for the construction of the filtration in Corollary 3.2. For an example of a complete variety for which the construction does not lead to a filtration by open subsets, see [14].

Remark 3.4.   One could construct the filtration in Corollary 3.2 also following the approach in [12, Section 5.2].

Applications to cohomology.   Recall that by work of Deligne [9], for every complex algebraic variety (assumed to be reduced, but possibly reducible), the cohomology $H^q(X, \mathbf{Q})$ and $H^q_c(X, \mathbf{Q})$ (singular cohomology and cohomology with compact support) carry natural (rational) mixed Hodge structures. A pure Hodge structure of weight q is *of Hodge–Tate type* if all Hodge numbers $h^{i,j}$ vanish, unless $i = j$. In particular, if the underlying vector space is nonzero, then q is even.

We can now give the main results concerning the cohomology of simplicial toric varieties that have convex, full-dimensional fan support.

Theorem 3.5.   *If X is a smooth, quasi-projective, complex toric variety, which has convex, full-dimensional fan support, then all maps $A_m(X) \to H^{\mathrm{BM}}_{2m}(X)$ are isomorphisms, where $A_m(X)$ is the Chow group of m-dimensional cycle classes and $H^{\mathrm{BM}}_{2m}(X)$ is the $(2m)$-th Borel–Moore homology group of X. If X is not smooth, but simplicial, then the maps are isomorphisms after tensoring with $\mathbb{Q}$.*

Proof.   We omit the proof, as it follows verbatim the one in [12, Section 5.2], using Proposition 3.2 $\square$

Theorem 3.6.   *Let X be a simplicial complex toric variety which has convex, full-dimensional fan support. For every q, the mixed Hodge structures on $H^q(X, \mathbb{Q})$ and $H^q_c(X, \mathbb{Q})$ are pure, of weight q, and of Hodge–Tate type. In particular, both $H^*(X, \mathbb{Q})$ and $H^*_c(X, \mathbb{Q})$ are even[1].*

We first prove the following lemma.

Lemma 3.7.   *Let X be a smooth, quasi-projective, complex toric variety, with convex, full-dimensional fan support. If $U_0 \subseteq \cdots \subseteq U_r = X$ is a filtration as in Corollary 3.2 and if $Z_i = X \smallsetminus U_i$, then the following hold:*

(a) *If q is odd and $0 \leq i \leq r$, then $H^q_c(Z_i, \mathbb{Q}) = 0$.*

(b) *If q is even and $0 \leq i \leq r - 1$, then we have an exact sequence*

$$0 \to H^q_c(Z_i \smallsetminus Z_{i+1}, \mathbb{Q}) \to H^q_c(Z_i, \mathbb{Q}) \to H^q_c(Z_{i+1}, \mathbb{Q}) \to 0.$$

In particular, for every q, the mixed Hodge structure on $H^q_c(X, \mathbb{Q})$ is pure, of weight q, and of Hodge–Tate type.

Proof.   We prove the assertions in (a) and (b) by descending induction on i. For $i = r$, the assertion in (a) is trivial. We now assume that (a) holds for $i + 1$ and show that both (a)

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[1]   We say that a graded vector space $\bigoplus_{i \in \mathbb{Z}} V^i$ is *even* if $V^i = 0$ for all odd i.

and (b) hold for i. Consider the long exact sequence for the cohomology with compact support

$$H_c^{q-1}(Z_{i+1}, \mathbb{Q}) \to H_c^q(Z_i \smallsetminus Z_{i+1}, \mathbb{Q}) \to H_c^q(Z_i, \mathbb{Q})$$
$$\to H_c^q(Z_{i+1}, \mathbb{Q}) \to H_c^{q+1}(Z_i \smallsetminus Z_{i+1}, \mathbb{Q}).$$

The key point is that by Corollary 3.2, $Z_i \smallsetminus Z_{i+1}$ is isomorphic to an affine space, hence $H_c^j(Z_i \smallsetminus Z_{i+1}, \mathbb{Q}) \simeq \mathbb{Q}$ if $j = 2 \cdot \dim(Z_i \smallsetminus Z_{i+1})$ and $H_c^j(Z_i \smallsetminus Z_{i+1}, \mathbb{Q}) = 0$, otherwise. The above exact sequence implies that $H_c^q(Z_i, \mathbb{Q}) = 0$ if q is odd and, using also the fact that (a) holds for $i + 1$, we see that the sequence in (b) is exact when q is even.

Note that the maps in the exact sequence in (b) are maps of mixed Hodge structures. In particular, they are strict with respect to both the weight and Hodge filtrations. We thus obtain by descending induction on i that the mixed Hodge structure on $H_c^q(Z_i, \mathbb{Q})$ is pure, of weight q, and of Hodge–Tate type (recall that $H_c^{2d}(\mathbf{A}^d)$ is pure, of weight $2d$, and of Hodge–Tate type). By taking $i = 0$, we obtain the last assertion in the lemma.   □

Proof of Theorem 3.6.   Note first that since X is simplicial, Poincaré duality implies that we have an isomorphism of mixed Hodge structures

$$H^q(X, \mathbb{Q}) \simeq H_c^{2d-q}(X, \mathbb{Q})^* \otimes \mathbf{Q}(-d),$$

where $d = \dim(X)$. This shows that the assertions about $H^*(X, \mathbb{Q})$ and $H_c^*(X, \mathbb{Q})$ are equivalent. Lemma 3.7 implies that the assertions about $H_c^*(X, \mathbb{Q})$ hold if X is smooth and quasi-projective, hence we are done in this case.

In the general case, it is enough to show that the assertions about $H^*(X, \mathbb{Q})$ hold. We use Chow's lemma and toric resolution of singularities (see Remark 2.9) to get a projective, birational toric map $g: \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and quasi-projective. Since the canonical map $g^*: H^q(X, \mathbb{Q}) \to H^q(\widetilde{X}, \mathbb{Q})$ is a morphism of mixed Hodge structures (hence strict with respect to both the weight and Hodge filtrations) and since we know that $H^q(\widetilde{X}, \mathbb{Q})$ is pure, of weight q, and of Hodge–Tate type, it is enough to show that g^* is injective.

In turn, injectivity follows easily from Poincaré duality on the $\mathbf{Q}$-manifold X, coupled with the projection formula [13, Section IX.3.7] ($\eta \in H^*(X, \mathbb{Q})$ and the equality holds in the Borel–Moore homology of X)

$$g_*(\{\widetilde{X}\} \cap g^*\eta) = (\{X\} \cap \eta).$$

This completes the proof of the theorem. We remark that the injectivity of g^* also follows from the easy-to-prove fact that, in our situation, $\mathbf{Q}_X$ is a direct summand of $Rg_*(\mathbf{Q}_{\widetilde{X}})$.   □

Remark 3.8.   The conclusions of Theorem 3.6 hold for X not necessarily simplicial, provided we replace cohomology (with compact supports) with intersection cohomology (with compact supports). This can be seen by applying the theorem to a toric resolution of X and by observing that the intersection cohomology of X is a natural subquotient of the cohomology of any resolution (see, for example, [6, Theorem 4.3.1]).

Betti numbers of simplicial toric varieties with convex, pure-dimensional fan support.   We are now ready to compute the Betti numbers of simplicial toric varieties that admit a proper map to an affine toric variety, of contractible type. This makes use, as in the complete case (see [12, Section 4.5]), of the Hodge–Deligne polynomial that we now briefly review.

Given a complex algebraic variety X of dimension n, one considers the mixed Hodge structure on the groups $H_c^i(X, \mathbf{Q})$. For every i and m, the m-th graded piece $\mathrm{gr}_m^W H_c^i(X, \mathbf{Q})$ with respect to the weight filtration is a pure Hodge structure of weight m. Therefore the Hodge numbers $h^{p,q}(\mathrm{gr}_m^W H_c^i(X, \mathbf{Q}))$ are defined whenever $p + q = m$. The Hodge–Deligne polynomial of X is given by

$$E(X; u, v) = \sum_{p,q \geq 0} e_{p,q} u^p v^q \in \mathbf{Z}[u, v],$$

where

$$e_{p,q} = \sum_{i=0}^{2n} (-1)^i h^{p,q}(\mathrm{gr}_{p+q}^W H_c^i(X, \mathbf{Q})).$$

In fact, the Hodge–Deligne polynomial is uniquely characterized by the following two properties:

(a) If X is smooth and projective, then $E(X; u, v)$ is the usual Hodge polynomial of X.

(b) If Y is a closed subset of X and $U = X \smallsetminus Y$, then

$$E(X; u, v) = E(Y; u, v) + E(U; u, v).$$

The Hodge–Deligne polynomial is also multiplicative, that is, if X and Y are complex algebraic varieties, then

$$E(X \times Y; u, v) = E(X; u, v) \cdot E(Y; u, v).$$

Due to the additivity property in b) above, it is easy to compute the Hodge–Deligne polynomial of varieties that can be written as disjoint unions of simple varieties, such as affine spaces or tori. Note that by property (a) above, we have $E(\mathbf{P}^1; u, v) = uv + 1$, hence property (b) implies $E(\mathbf{A}^1; u, v) = uv$ and $E(\mathbf{C}^*; u, v) = uv - 1$. Using the fact that the Hodge–Deligne polynomial is multiplicative, one obtains $E(\mathbf{A}^n; u, v) = (uv)^n$ and $E((\mathbf{C}^*)^n; u, v) = (uv - 1)^n$.

If X is smooth and projective, of dimension n, then

$$E(X; t, t) = \sum_{i=0}^{2n} (-1)^i b_i(X) t^i,$$

where

$$b_i(X) = \dim_{\mathbf{Q}} H^i(X, \mathbf{Q}),$$

that is, $E(X; t, t) = P_X(-t)$, where $P_X(t)$ is the Poincaré polynomial. This is a consequence of (a) and of the Hodge decomposition. However, for an arbitrary X, we cannot recover $\dim_{\mathbf{Q}} H_c^i(X, \mathbf{Q})$ from the Hodge–Deligne polynomial. One case when this can be done is when we know that the mixed Hodge structure on each $H_c^i(X, \mathbf{Q})$ is pure of weight i. In this case, it follows from the definition that

$$E(X; u, v) = \sum_{p,q \geq 0} (-1)^{p+q} h^{p,q}(H_c^{p+q}(X, \mathbf{Q})) u^p v^q$$

and therefore that

$$E(X; t, t) = \sum_{i=0}^{2n} (-1)^i \dim_{\mathbf{Q}} H_c^i(X, \mathbf{Q}) t^i.$$

Remark 3.9. The formalism of the Hodge–Deligne polynomial extends to constructible complexes $K^\bullet$ of sheaves with the property that the stalks of their cohomology sheaves carry a mixed Hodge structure, or just a weight filtration. In the proof of Theorem 7.6 below we will use this with $K^\bullet$ the restriction of the intersection complex IC_X of a toric variety X to a union of torus orbits.

Let X be an n-dimensional toric variety. Let us denote by $d_\ell(X)$ the number of cones in Δ_X of codimension ℓ. In other words, $(d_n(X), \ldots, d_0(X))$ is the f-vector of X (see [20, 21]) (the reason we label the entries differently from the usual way is for convenience in the relative setting). Since a toric variety X is the disjoint union of its orbits, we obtain the following well-known formula

$$(3.4) \qquad E(X; u, v) = \sum_{\sigma \in \Delta_X} (uv - 1)^{\mathrm{codim}(\sigma)} = \sum_{i=0}^{n} d_i(X) \cdot (uv - 1)^i.$$

Corollary 3.10. *Let X be a simplicial toric variety of dimension n, with convex, full-dimensional fan support. The odd cohomology (ordinary and with compact supports) vanishes and we have that*

$$\dim_\mathbb{Q} H_c^{2m}(X, \mathbb{Q}) = \dim_\mathbb{Q} H^{2n-2m}(X, \mathbb{Q}) = \sum_{i=m}^{n} (-1)^{i-m} \binom{i}{m} d_i(X).$$

Proof. By Theorem 3.6, the mixed Hodge structure on each $H_c^i(X, \mathbb{Q})$ is pure of weight i, hence the formula for the dimension of the cohomology with compact support follows from (3.4). The formula for the dimension of the usual cohomology then follows by Poincaré duality. □

4. Cohomology of fibers of toric maps with simplicial source

Our goal in this section is to compute the Betti numbers of the fibers of toric fibrations $f \colon X \to Y$, when X is simplicial. As in the previous section, the main ingredient is the following purity result.

Theorem 4.1. *Let $f \colon X \to Y$ be a proper toric map between toric varieties, with X simplicial. For every $y \in Y$ and every q, the mixed Hodge structure on $H^q(f^{-1}(y), \mathbb{Q})$ is pure, of weight q, and of Hodge–Tate type. In particular, $H^*(f^{-1}(y), \mathbb{Q})$ is even.*

We will deduce the theorem from Theorem 3.6, by showing that if $f \colon X \to Y$ is a toric fibration to an affine toric variety of contractible type, then the restriction map in cohomology from X to the fiber over the torus-fixed point of Y is an isomorphism. In what follows, we prove this in a slightly more general setting that we will need in the next section.

Suppose that X is a complex algebraic variety, considered with the analytic topology. Let G be an algebraic group acting on X and let act$\colon G \times X \to X$ be the corresponding morphism. By an *equivariant complex of sheaves*[2] on X we mean a complex of sheaves $\mathcal{E}$ on X

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[2] The sheaves are assumed to be sheaves of $\mathbb{Q}$-vector spaces, with respect to the analytic topology.

with the property that there is an isomorphism[3] $\mathrm{act}^*(\mathcal{E}) \simeq \mathrm{pr}_2^*(\mathcal{E})$ of complexes on $G \times X$. With slight abuse of language, we say that an object in the derived category of sheaves is equivariant if it can be represented by an equivariant complex. An important example is that of the constant sheaf $\mathbb{Q}_X$. It is easy to see that if $f \colon X \to Y$ is a G-equivariant morphism and $\mathcal{E}$ is equivariant on X, then $Rf_*(\mathcal{E})$ is equivariant on Y.

The following lemma was proved in [10, Lemma 6.5] in the case of the intersection complex. The general case follows in the same way, but we reproduce the argument for the benefit of the reader.

Lemma 4.2. *Let $Y = U_\sigma$ be an affine toric variety with fixed point y. If $v \in N_Y \cap \mathrm{Int}(\sigma)$ and we consider the action of $\mathbf{C}^*$ on Y induced by the one-parameter subgroup γ_v, then for every $\mathbf{C}^*$-equivariant complex $\mathcal{E}$ on Y, the natural graded map $H^*(Y, \mathcal{E}) \to H^*(\mathcal{E}_y)$ is an isomorphism.*

Proof. The assertion in the lemma is equivalent to the fact that $H^*(Y, j_!(\mathcal{E})) = 0$, where $j \colon Y_0 = Y \smallsetminus \{y\} \hookrightarrow Y$ is the inclusion. The hypothesis on v implies that the map $\mathbf{C}^* \times Y \to Y$ given by $(t, x) \to \gamma_v(t) \cdot x$ extends to a map $h \colon \mathbf{A}^1 \times Y \to Y$ such that

$$h^{-1}(y) = (\mathbf{A}^1 \times \{y\}) \cup (\{0\} \times Y).$$

Consider the morphism $g \colon Y \to \mathbf{A}^1 \times Y$ given by $g(x) = (1, x)$, so that $h \circ g$ is an isomorphism. Therefore the composition

$$H^*(Y, j_!(\mathcal{E})) \to H^*(\mathbf{A}^1 \times Y, h^* j_!(\mathcal{E})) \to H^*(Y, g^* h^* j_!(\mathcal{E}))$$

is an isomorphism, hence it is enough to prove that $H^*(\mathbf{A}^1 \times Y, h^* j_!(\mathcal{E})) = 0$. On the other hand, we have $h^{-1}(Y_0) = \mathbf{C}^* \times Y_0$ and it is easy to see that

$$h^* j_!(\mathcal{E}) \simeq j_!' h_0^*(\mathcal{E}),$$

where $j' \colon \mathbf{C}^* \times Y_0 \hookrightarrow \mathbf{A}^1 \times Y$ is the inclusion and $h_0 \colon \mathbf{C}^* \times Y_0 \to Y_0$ is induced by h. Since $\mathcal{E}$ is equivariant with respect to the $\mathbf{C}^*$-action, it follows that we have an isomorphism

$$h_0^*(\mathcal{E}) \simeq \mathrm{pr}_2^*(\mathcal{E}),$$

hence

$$j_!' h_0^*(\mathcal{E}) \simeq j_!'' (\mathbb{Q}_{\mathbf{C}^*} \boxtimes j_!(\mathcal{E})),$$

where $j'' \colon \mathbf{C}^* \hookrightarrow \mathbf{C}$ is the inclusion. Since $H^*(\mathbf{A}^1, j_!''(\mathbf{Q}_{\mathbf{C}_*})) = 0$, the Künneth formula implies $H^*(\mathbf{A}^1 \times Y, h^* j_!(\mathcal{E})) = 0$. □

Remark 4.3. Suppose that $f \colon X \to Y$ is a proper surjective toric map, where Y is as in Lemma 4.2. If $\mathcal{E}$ is a T_X-equivariant complex on X, then we may choose v as in the lemma such that v lies in the image of f_N. Therefore we get actions of $\mathbf{C}^*$ on X and Y such that f is $\mathbf{C}^*$-equivariant. In this case we may apply the lemma to the complex $Rf_*(\mathcal{E})$ and conclude that we have canonical isomorphisms

$$H^i(X, \mathcal{E}) \simeq H^i(Y, Rf_*(\mathcal{E})) \simeq H^i(Rf_*(\mathcal{E})_y) \simeq H^i(f^{-1}(y), \mathcal{E}),$$

where the last isomorphism follows from proper base-change.

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[3] The usual definition also requires the isomorphism to satisfy a cocycle condition. However, we do not need this extra condition.

We include the next remark for future reference.

Remark 4.4. In the setting of Lemma 4.2, we may take $\mathcal{E} = \mathcal{I}_Y := \mathrm{IC}_Y[-\dim(Y)]$, where IC_Y is the intersection cohomology complex on Y to conclude

$$IH^i(Y, \mathbb{Q}) \simeq \mathcal{H}^i(\mathcal{I}_Y)_y.$$

If $f : X \to Y$ is a proper, surjective, toric map, then we may take $\mathcal{E} = \mathcal{I}_X$ to conclude

$$IH^i(X, \mathbb{Q}) \simeq H^i(f^{-1}(y), \mathcal{I}_X).$$

We can now prove the main result of this section.

Proof of Theorem 4.1. We first note that we may assume that f is a fibration, Y is affine and of contractible type, and $y \in Y$ is the fixed point. Indeed, it follows from Proposition 2.1 and Remark 2.3 that we may write f as a composition

$$X \xrightarrow{g} Z \xrightarrow{h} Y,$$

with g a fibration and h finite. Since every fiber of f is a disjoint union of fibers of g, it follows that after replacing f by g, we may assume that f is a fibration. Suppose now that $\tau \in \Delta_Y$ is such that $y \in O(\tau)$. After replacing f by $f^{-1}(U_\tau) \to U_\tau$, we may assume that $Y = U_\tau$. Furthermore, by Lemma 2.6, we may assume that the linear span of τ is the ambient space and that $y = x_\tau$.

In this case we may apply Lemma 4.2 to $Rf_*(\mathbb{Q}_X)$ (see Remark 4.3) to conclude that the canonical restriction map

$$H^i(X, \mathbb{Q}) \to H^i(f^{-1}(y), \mathbb{Q})$$

is an isomorphism. Since this is a morphism of mixed Hodge structures, the assertions in the theorem follow from those in Theorem 3.6. □

As in the case of toric varieties, it is easy to compute the Hodge–Deligne polynomial of fibers of toric fibrations. Given a toric fibration $f : X \to Y$ and a cone $\tau \in \Delta_Y$, we put

$$d_\ell(X/\tau) = \#\{\sigma \in \Delta_X : f_*(\sigma) = \tau, \ \mathrm{codim}(\sigma) - \mathrm{codim}(\tau) = \ell\}.$$

Remark 4.5. Note that when Y is a point, then $d_\ell(X/\{0\})$ is the same as $d_\ell(X)$, as introduced in the previous section. Therefore, if the relative dimension of f is

$$r = \dim(X) - \dim(Y),$$

then the vector $(d_r(X/\tau), \ldots, d_0(X/\tau))$ is a relative version of the f-vector of a toric variety.

Proposition 4.6. *Let X and Y be toric varieties and let $f : X \to Y$ be a toric fibration. If τ is a cone in Δ_Y and $y \in O(\tau)$, then*

$$E(f^{-1}(y); u, v) = \sum_{\ell \geq 0} d_\ell(X/\tau) \cdot (uv - 1)^\ell.$$

In particular, we have $\chi(f^{-1}(y)) = d_0(X/\tau)$.

Proof.   Recall that by Proposition 2.7, we can write $f^{-1}(y)$ as a disjoint union of locally closed subsets, with the subsets in one-to-one correspondence with the cones $\sigma \in \Delta_X$ that satisfy $f_*(\sigma) = \tau$. Moreover, the subset corresponding to σ is isomorphic to $(\mathbf{C}^*)^{\mathrm{codim}(\sigma)-\mathrm{codim}(\tau)}$. By additivity of the Hodge–Deligne polynomial, we thus get

$$E(f^{-1}(y); u, v) = \sum_{f_*(\sigma)=\tau} (uv - 1)^{\mathrm{codim}(\sigma)-\mathrm{codim}(\tau)} = \sum_{\ell \geq 0} d_\ell(X/\tau) \cdot (uv - 1)^\ell.$$

The last assertion follows from the usual identity $\chi(f^{-1}(y)) = E(f^{-1}(y); 1, 1)$.   □

Corollary 4.7.   *Let X and Y be toric varieties, with X simplicial, and let $f : X \to Y$ be a toric fibration. If τ is a cone in Δ_Y and $y \in O(\tau)$, then*

$$\dim_{\mathbf{Q}} H^{2m}(f^{-1}(y), \mathbf{Q}) = \sum_{\ell \geq m}(-1)^{\ell-m}\binom{\ell}{m} d_\ell(X/\tau).$$

Proof.   Note that since f is proper, $f^{-1}(y)$ is compact, hence the usual cohomology agrees with the cohomology with compact support. By Theorem 4.1, the mixed Hodge structure on $H^i(f^{-1}(y), \mathbf{Q})$ is pure, hence

$$E(f^{-1}(y); t, t) = \sum_{i \geq 0}(-1)^i b_i(f^{-1}(y))t^i,$$

where

$$b_i(f^{-1}(y)) = \dim_{\mathbf{Q}} H^i(f^{-1}(y), \mathbf{Q}).$$

On the other hand, it follows from Proposition 4.6 that

$$E(f^{-1}(y); t, t) = \sum_{\ell \geq 0} d_\ell(X/\tau)(t^2 - 1)^\ell$$

$$= \sum_{\ell \geq 0} d_\ell(X/\tau) \cdot \sum_{m=0}^{\ell}(-1)^{\ell-m}\binom{\ell}{m} t^{2m}$$

$$= \sum_{m \geq 0}\left(\sum_{\ell \geq m}(-1)^{\ell-m}\binom{\ell}{m} d_\ell(X/\tau)\right) t^{2m}.$$

The formula in the corollary now follows by equating the coefficients of the powers of t in the two expressions for $E(f^{-1}(y); t, t)$.   □

Remark 4.8.   If in Corollary 4.7 we consider the case when Y is a point, then we recover the familiar formula for the Betti numbers of a simplicial, complete toric variety (see [12, Section 4.5]).

Remark 4.9.   In fact, one can prove the assertions in Theorems 3.6 and 4.1 without making use of the filtration constructed in the previous section, arguing as follows. We have seen that it is enough to prove that when $f : X \to Y$ is a proper toric fibration, with X smooth and Y affine, of contractible type, and with $y \in Y$ being the torus-fixed point, then the mixed Hodge structures on $H^i(X, \mathbb{Q})$ and $H^i(f^{-1}(y), \mathbb{Q})$ are pure of weight i (the fact that the Hodge structures are of Hodge–Tate type then follows from the computation of the Hodge–Deligne

polynomials). As we have seen, we have an isomorphism of mixed Hodge structures

$$H^i(X, \mathbb{Q}) \to H^i(f^{-1}(y), \mathbb{Q}).$$

Since X is smooth, all weights on $H^i(X, \mathbb{Q})$ are $\geq i$, while since $f^{-1}(y)$ is compact, all weights on $H^i(f^{-1}(y), \mathbb{Q})$ are $\leq i$. We conclude that the mixed Hodge structures on each of $H^i(X, \mathbb{Q})$ and $H^i(f^{-1}(y), \mathbb{Q})$ are pure of weight i.

5. The Decomposition Theorem for toric maps

The main result of this section is the following version of the Decomposition Theorem in the case of toric fibrations. Recall that for an algebraic variety X, we denote by IC_X the intersection complex on X.

Theorem 5.1. *If X and Y are complex toric varieties and $f : X \to Y$ is a toric fibration, then we have a decomposition*

$$(5.1) \qquad\qquad Rf_*(\mathrm{IC}_X) \simeq \bigoplus_{\tau \in \Delta_Y} \bigoplus_{b \in \mathbb{Z}} \mathrm{IC}_{V(\tau)}^{\oplus s_{\tau,b}}[-b].$$

Furthermore, the nonnegative integers $s_{\tau,b}$ satisfy the following conditions:

(i) $s_{\tau,b} = s_{\tau,-b}$ *for every $\tau \in \Delta_Y$ and every $b \in \mathbb{Z}$.*

(ii) *If f is projective, then $s_{\tau,b} \geq s_{\tau,b+2\ell}$ for every $\tau \in \Delta_Y$ and every $b, \ell \in \mathbb{Z}_{\geq 0}$.*

(iii) $s_{\tau,b} = 0$ *if $b + \dim(X) - \dim(V(\tau))$ is odd.*

With the notation in Theorem 5.1, for every $\tau \in \Delta_Y$ we put $\delta_\tau = \sum_b s_{\tau,b}$. It is clear that δ_τ is a nonnegative integer. The subvariety $V(\tau)$ is a *support* for f if $\delta_\tau > 0$. In Sections 6 and 7 we will give combinatorial descriptions of the invariants δ_τ, determining in particular the supports of f.

Remark 5.2. Suppose that X and Y are toric varieties and $f : X \to Y$ is any proper toric map. It follows from Proposition 2.1 and Remark 2.3 that we can factor f as

$$X \xrightarrow{g} Z \xrightarrow{h} Y,$$

with g a toric fibration and h a finite toric map. For every $\tau \in \Delta_Z$, we have an induced finite morphism of algebraic groups $O(\tau) \to O(h_*(\tau))$. If we denote by $O'(\tau)$ the image of this map, then $O'(\tau)$ is a torus and we have a finite, surjective, étale map $h_\tau : O(\tau) \to O'(\tau)$, which is the quotient by a finite group (see Remark 2.4). It follows that if $\mathscr{L}_\tau := (h_\tau)_*(\mathbf{Q}_{O(\tau)})$, then $\mathscr{L}_\tau$ is a local system on $O'(\tau)$. Moreover, the induced map $\overline{h}_\tau : V(\tau) \to \overline{O'(\tau)}$ is finite, hence the direct image by this map is t-exact for the middle perversity t-structure, and therefore it preserves intersection complexes with twisted coefficients. This implies that

$$\overline{h}_{\tau*}(\mathrm{IC}_{V(\tau)}) \simeq \mathrm{IC}_{\overline{O'(\tau)}}(\mathscr{L}_\tau).$$

We thus obtain from Theorem 5.1 the following decomposition in this general toric setting:

$$Rf_*(\mathrm{IC}_X) \simeq \bigoplus_{\tau \in \Delta_Z} \bigoplus_{b \in \mathbb{Z}} \mathrm{IC}_{\overline{O'(\tau)}}^{\oplus s_{\tau,b}}(\mathscr{L}_\tau)[-b].$$

Clearly, the supports of f are the images via h of the supports of the toric fibration g.

Before giving the proof of Theorem 5.1 we make some preparations. We begin by recalling the following general statement of the Decomposition Theorem, see [1, Théorème 6.2.5].

Theorem 5.3. *Let X and Y be complex algebraic varieties and consider a proper morphism $f: X \to Y$. We have a finite direct sum decomposition*

$$Rf_*(\mathrm{IC}_X) \simeq \bigoplus_\alpha \mathrm{IC}_{\overline{Y_\alpha}}(L_\alpha)[-d_\alpha],$$

where each Y_α is a smooth, irreducible, locally closed subset of Y, L_α is a local system on Y_α and $d_\alpha \in \mathbb{Z}$.

In order to prove Theorem 5.1, we will need to show that under the assumptions of the theorem, all varieties $\overline{Y_\alpha}$ are torus-invariant and the local systems L_α are trivial. Properties (i) and (ii) will then follow from Poincaré duality and relative Hard Lefschetz. Finally, property (iii) will be a consequence of the following proposition.

Proposition 5.4. *If $f: X \to Y$ is a proper toric map, then for every $y \in Y$, we have $H^i(f^{-1}(y), \mathrm{IC}_X) = 0$ whenever $i + \dim(X)$ is odd.*

Proof. By Proposition 2.1 and Remark 2.3, we may factor f as a composition

$$X \xrightarrow{g} Z \xrightarrow{h} Y,$$

with g a toric fibration and h finite. Since a fiber of f is a disjoint union of fibers of g, it follows that we may assume that f is a fibration. Let $\pi: X' \to X$ be a proper, birational, toric morphism such that X' is smooth and let $f' = f \circ \pi$. Since π is birational, it follows from Theorem 5.3 that IC_X is a direct summand of $R\pi_*(\mathrm{IC}_{X'})$. By restricting over $f^{-1}(y)$, applying base-change, and taking the i-th cohomology, we conclude that $H^i(f^{-1}(y), \mathrm{IC}_X)$ is a summand of

$$H^i(f^{-1}(y), R\pi_*(\mathrm{IC}_{X'})) = H^i(f'^{-1}(y), \mathrm{IC}_{X'}) = H^{i+n}(f'^{-1}(y), \mathbb{Q}),$$

where $n = \dim(X)$ and the second equality follows from the fact that X' being smooth, we have $\mathrm{IC}_{X'} = \mathbb{Q}_{X'}[n]$. On the other hand, since X' is smooth and $f \circ \pi$ is a fibration, we may apply Theorem 4.1 to get $H^{i+n}(f'^{-1}(y), \mathbb{Q}) = 0$ when $i + n$ is odd. This completes the proof of the proposition.   □

Remark 5.5. It follows from Saito's theory of mixed Hodge modules [19] that for every proper morphism $f: X \to Y$ of complex algebraic varieties and every $y \in Y$, the cohomology groups $H^{i-\dim X}(f^{-1}(y), \mathrm{IC}_X) = H^i(f^{-1}(y), \mathcal{I}_X)$ carry a natural mixed Hodge structure. The argument in the proof of Proposition 5.4 together with Theorem 4.1 imply that if f is a toric map, then this mixed Hodge structure is pure of weight i.

The fact that in the toric decomposition theorem we only have torus-invariant subvarieties as supports will follow from the next lemma. Given a toric variety Y, we denote by Ω_Y the stratification by the T_Y-orbits. Recall that a complex of sheaves $\mathcal{E}$ on Y is Ω_Y-*constructible* if for every i and every $O \in \Omega_Y$, the restriction $\mathcal{H}^i(\mathcal{E})|_O$ is a local system on O. The intersection complex of a toric variety Y is Ω_Y-constructible in a strong sense, i.e. each of these restrictions

is a constant (torus) equivariant sheaf on the orbit (see [2, Lemma 5.15]). The same is true for the direct image complex in (5.1), and the lemma that follows is a step in proving Theorem 5.1.

Lemma 5.6. *If $f: X \to Y$ is a proper toric fibration, then $Rf_*(\mathrm{IC}_X)$ is Ω_Y-constructible. In fact, the restriction of each $\mathcal{H}^i(Rf_*(\mathrm{IC}_X))$ to a T_Y-orbit is a constant sheaf. In particular, IC_Y is Ω_Y-constructible.*

Proof.   Let $\tau \in \Delta_Y$ be fixed. In order to describe the restriction of some $\mathcal{H}^i(Rf_*(\mathrm{IC}_X))$ to the orbit $O(\tau)$, we may restrict to the affine open subset U_τ and thus assume that $Y = U_\tau$. Lemma 2.6 implies that we have isomorphisms $Y \simeq Y' \times O(\tau)$ and $X \simeq X' \times O(\tau)$, with Y' having a fixed point y, such that f gets identified to $f' \times \mathrm{Id}_{O(\tau)}$, where $f': X' \to Y'$ is a toric fibration. In this case IC_X gets identified to $\mathrm{pr}_1^*(\mathrm{IC}_{X'})[r]$, where $r = \dim(O(\tau))$. It is then clear that

$$\mathcal{H}^i(Rf_*(\mathrm{IC}_X))|_{O(\tau)} \simeq \mathcal{H}^{i+r}(Rf'_*(\mathrm{IC}_{X'}))_y \otimes \mathbb{Q}_{O(\tau)}.$$

This gives the first two assertions in the lemma. The third is the special case $f = \mathrm{Id}$. □

We can now prove the main result of this section.

Proof of Theorem 5.1.   Given a toric fibration $f: X \to Y$, we obtain via Theorem 5.3 a finite direct sum decomposition

$$Rf_*(\mathrm{IC}_X) \simeq \bigoplus_\alpha \mathrm{IC}_{\overline{Y_\alpha}}(L_\alpha)[-d_\alpha],$$

with each Y_α a smooth, irreducible, locally closed subset of Y and L_α a local system on Y_α. We may assume that each L_α is nonzero and indecomposable.

Since $Rf_*(\mathrm{IC}_X)$ is Ω_Y-constructible by Lemma 5.6, so is each term $\mathrm{IC}_{\overline{Y_\alpha}}(L_\alpha)$. It follows that $\overline{Y_\alpha} = V(\sigma_\alpha)$ for a unique cone $\sigma_\alpha \in \Delta_Y$ and that we can take $Y_\alpha = O(\sigma_\alpha)$.

The proof of the same lemma implies that the restriction of $\mathcal{H}^{-\dim O(\sigma_\alpha)}(\mathrm{IC}_{\overline{V(\sigma_\alpha)}}(L_\alpha))$ to $O(\sigma_\alpha)$ is constant. This restriction is L_α, which is thus constant, hence isomorphic to $\mathbb{Q}_{O(\sigma_\alpha)}$ (being indecomposable). Therefore $\mathrm{IC}_{\overline{Y_\alpha}}(L_\alpha) = \mathrm{IC}_{V(\sigma_\alpha)}$.

The assertions in (i) and (ii) follow from Poincaré duality and relative Hard Lefschetz (see [1, Théorème 5.4.10]). In order to check the assertion in (iii), we consider for $\tau \in \Delta_Y$ the stalk at x_τ for both sides of (5.1). By taking the i-th cohomology and applying base-change, we conclude that $\mathcal{H}^{i-b}(\mathrm{IC}_{V(\tau)})_{x_\tau}^{\oplus s_{\tau,b}}$ is a direct summand of $H^i(f^{-1}(x_\tau), \mathrm{IC}_X)$. If we have $b + \dim(X) - \dim(V(\tau))$ odd, then Lemma 5.4 implies that $H^i(f^{-1}(x_\tau), \mathrm{IC}_X) = 0$ when $i = b - \dim(V(\tau))$. However, in this case $\mathcal{H}^{i-b}(\mathrm{IC}_{V(\tau)})_{x(\tau)} \simeq \mathbb{Q}$, which implies $s_{\tau,b} = 0$. □

Remark 5.7.   As we have mentioned in the Introduction, Katz and Stapledon associate some invariants, *local h-polynomials* to certain maps between posets. This is done in a purely combinatorial way. Given a toric fibration $f: X \to Y$, their framework can be applied to the map $f_*: \Delta_X \to \Delta_Y$ and it turns out that our invariants $s_{\tau,b}$ can be interpreted as coefficients of the local h-polynomial. Due to the combinatorial definition, the nonnegativity of these coefficients is not a priori clear. However, one of the main results in [15] says that in the case of a regular rational polyhedral subdivision of a rational polytope (which corresponds to a projective toric birational morphism between toric varieties), the coefficients of the local h-polynomial are nonnegative, symmetric, and unimodal (see [15, Theorem 6.1]).

6. Combinatorics of the toric Decomposition Theorem: The simplicial case

In this section we study toric fibrations $f: X \to Y$, with X and Y simplicial toric varieties. In this case we can determine explicitly all multiplicities $s_{\tau,b}$ that appear in Theorem 5.1.

We begin by recalling some elementary combinatorial definitions and facts about the incidence algebra associated with a partially ordered set. For an introduction to incidence algebras and applications, see [18] or [23, Section 3.6]. If $(\mathcal{P}, \leq)$ is a finite poset and K is a commutative ring with identity, we have the *incidence algebra* $\mathbb{I}(\mathcal{P}, K)$ consisting of the set of functions

$$f: \{(a, b) \in \mathcal{P} \times \mathcal{P} : a \leq b\} \to K$$

with a *convolution* operation defined by

$$f \star g(a, b) := \sum_{a \leq c \leq b} f(a, c) \cdot g(c, b).$$

This is an associative operation with an identity element given by the *delta function* δ[4], with $\delta(a, b) = 1$ if $a = b$ and $\delta(a, b) = 0$, otherwise. It is easy to check, see [23, Proposition 3.6.2], that f has a (unique) inverse with respect to convolution if and only if $f(a, a)$ is a unit of K for every $a \in \mathcal{P}$ (the proof in [23], given for $K = \mathbb{Z}$, holds without any change for any commutative ring with identity).

An important example is that of the function ζ given by $\zeta(a, b) = 1$ whenever $a \leq b$. The inverse of ζ with respect to convolution is the *Möbius function* $\mu_{\mathcal{P}}$ of $\mathcal{P}$. For example, if $(\mathcal{P}, \leq)$ is the set of all subsets of a finite set, ordered by inclusion, then it is easy to see that $\mu_{\mathcal{P}}(A, B) = (-1)^{\#(B \smallsetminus A)}$ whenever $A \subseteq B$.

Another fact that follows easily from definition is that if f is a function as above with inverse g with respect to convolution and $\phi, \psi: \mathcal{P} \to K$ are functions such that

$$\phi(x) = \sum_{y \leq x} f(y, x) \psi(y) \quad \text{for every } x \in \mathcal{P},$$

then

$$\psi(y) = \sum_{x \leq y} g(x, y) \phi(x) \quad \text{for every } x \in \mathcal{P}.$$

When $f = \delta$, this is the *Möbius inversion formula*.

After these preparations, we return to toric maps. In the following theorem we consider a toric fibration $f: X \to Y$, in which both X and Y are simplicial. In this case the decomposition (5.1) becomes easier to describe, since

$$IC_X = \mathbf{Q}_X[\dim(X)] \quad \text{and} \quad IC_{V(\tau)} = \mathbb{Q}_{V(\tau)}[\dim(V(\tau))]$$

for every cone $\tau \in \Delta_Y$ (we use the fact that both X and $V(\tau)$ are $\mathbb{Q}$-manifolds). Recall that if $f: X \to Y$ is a toric fibration, then for every $\tau \in \Delta_Y$ we put

$$d_\ell(X/\tau) = \#\{\alpha \in \Delta_X : f_*(\alpha) = \tau, \text{codim}(\alpha) - \text{codim}(\tau) = \ell\}.$$

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[4] This should not be confused with our key invariant δ, a function of a single variable.

Theorem 6.1. *Suppose we are in the setting of Theorem 5.1, with both X and Y simplicial.*

(i) *For every $\tau \in \Delta_Y$, we have*

$$\delta_\tau := \sum_b s_{\tau,b} = \sum_{\sigma \subseteq \tau} (-1)^{\dim(\tau)-\dim(\sigma)} d_0(X/\sigma).$$

(ii) *For every $m \in \mathbb{Z}$ and every $\tau \in \Delta_Y$, we have*

$$s_{\tau,2m+\dim(V(\tau))-\dim(X)} = \sum_{\sigma \subseteq \tau} (-1)^{\dim(\tau)-\dim(\sigma)} \cdot \sum_{\ell \geq m} (-1)^{\ell-m} \binom{\ell}{m} d_\ell(X/\sigma)$$

(where the right-hand side is understood to be 0 if $m < 0$) while $s_{\tau,i+\dim(V(\tau))-\dim(X)} = 0$ if i is odd.

Proof. Let $d_X = \dim(X)$. Since X and Y are simplicial, it follows from Theorem 5.1 that we have a decomposition

$$(6.1) \qquad Rf_*(\mathbb{Q}_X[d_X]) \simeq \bigoplus_{\tau \in \Delta_Y} \bigoplus_{b \in \mathbb{Z}} \mathbb{Q}_{V(\tau)}^{\oplus s_{\tau,b}} [\dim(V(\tau)) - b].$$

If $\sigma \in \Delta_Y$, by taking the stalk at x_σ and computing the $(i - d_X)$-th cohomology, we obtain via base-change

$$(6.2) \qquad H^i(f^{-1}(x_\sigma), \mathbb{Q}) \simeq \bigoplus_{\tau \subseteq \sigma} \mathbb{Q}^{\oplus s_{\tau,i+\dim(V(\tau))-d_X}}.$$

The second assertion in (ii) follows directly from Theorem 5.1, while Theorem 4.1 implies $H^j(f^{-1}(x_\sigma), \mathbb{Q}) = 0$ for j odd. We conclude that

$$\chi(f^{-1}(x_\sigma)) = \sum_{i \geq 0} \dim_\mathbb{Q} H^i(f^{-1}(y), \mathbb{Q}) = \sum_{\tau \subseteq \sigma} \delta_\tau.$$

The Möbius inversion formula for the poset Δ_Y implies

$$(6.3) \qquad \delta_\tau = \sum_{\sigma \subseteq \tau} \mu_{\Delta_Y}(\sigma, \tau) \chi(f^{-1}(x_\sigma)) = \sum_{\sigma \subseteq \tau} \mu_{\Delta_Y}(\sigma, \tau) d_0(X/\sigma),$$

where the second equality follows from Proposition 4.6. On the other hand, $\mu_{\Delta_Y}(\sigma, \tau)$ only depends on the interval $[\sigma, \tau]$ in Δ_Y and since τ is simplicial, this interval is in order-preserving bijection with the poset of all subsets of a set with $\dim(\tau) - \dim(\sigma)$ elements. Therefore $\mu_{\Delta_Y}(\sigma, \tau) = (-1)^{\dim(\tau)-\dim(\sigma)}$. The formula in (6.3) thus gives the assertion in (i).

We proceed similarly to prove (ii). Let m be a fixed integer. It follows from (6.2) that for every $\sigma \in \Delta_Y$ we have

$$\dim_\mathbb{Q} H^{2m}(f^{-1}(x_\sigma), \mathbb{Q}) = \sum_{\tau \subseteq \sigma} s_{\tau,2m+\dim(V(\tau))-d_X}.$$

The Möbius inversion formula and Corollary 4.7 imply

$$s_{\tau,2m+\dim(V(\tau))-d_X} = \sum_{\sigma \subseteq \tau} (-1)^{\dim(\tau)-\dim(\sigma)} \dim_\mathbb{Q} H^{2m}(f^{-1}(x_\sigma), \mathbb{Q})$$

$$= \sum_{\sigma \subseteq \tau} (-1)^{\dim(\tau)-\dim(\sigma)} \cdot \sum_{\ell \geq m} (-1)^{\ell-m} \binom{\ell}{m} d_\ell(X/\sigma).$$

This completes the proof of the theorem.   □

Remark 6.2. The reader can compare the formula for the invariants $s_{\tau,b}$ in Theorem 6.1 with the formula in [15, Lemma 4.12] for the local h-polynomial of a map of posets $\Gamma \to B$, in which Γ is simplicial and B is a Boolean algebra.

Remark 6.3. Let $f \colon X \to Y$ be a toric fibration, with both X and Y simplicial. It follows from Theorem 6.1 that for every $\tau \in \Delta_Y$, the expression

$$\sum_{\sigma \subseteq \tau} (-1)^{\dim(\tau)-\dim(\sigma)} d_0(X/\sigma)$$

is nonnegative (and it is positive if and only if $V(\tau)$ is a support for f). We do not know a direct combinatorial argument that would imply that the expression is nonnegative. A similar remark can be made in the not-necessarily-simplicial case, following the combination of Theorems 7.3 and 7.6. In Remark 6.5 below we give such an argument when f is birational between simplicial toric varieties and $\dim(\tau) \leq 3$.

Remark 6.4. Note that the invariants $s_{\tau,b}$ in Theorem 5.1 satisfy conditions (i) and (ii) coming from Poincaré duality and relative Hard Lefschetz. In the setting of Theorem 6.1, these translate into interesting conditions satisfied by the invariants $d_\ell(X/\tau)$, for the cones $\tau \in \Delta_Y$. More precisely, suppose that $f \colon X \to Y$ is a projective toric fibration between simplicial toric varieties. For a cone $\sigma \in \Delta_Y$ and $m \geq 0$, let us put

$$\widetilde{d}_m(X/\sigma) = \sum_{\ell \geq m} (-1)^{\ell-m} \binom{\ell}{m} d_\ell(X/\sigma).$$

Recall that by Corollary 4.7, we have

$$\widetilde{d}_m(X/\sigma) = \dim_{\mathbb{Q}} H^{2m}(f^{-1}(y); \mathbb{Q}) \quad \text{for any } y \in O(\sigma).$$

In particular, we have $\widetilde{d}_m(X/\sigma) \geq 0$. With this notation, Poincaré duality says that for every m with $0 \leq m \leq \dim(X) - \dim(V(\tau))$, if $m' = \dim(X) - \dim(V(\tau)) - m$, then

$$\sum_{\sigma \subseteq \tau} (-1)^{\dim(\tau)-\dim(\sigma)} \widetilde{d}_m(X/\sigma) = \sum_{\sigma \subseteq \tau} (-1)^{\dim(\tau)-\dim(\sigma)} \widetilde{d}_{m'}(X/\sigma).$$

Similarly, relative Hard Lefschetz says that if $0 \leq m \leq \frac{1}{2}(\dim(X) - \dim(V(\tau)))$, then

$$\sum_{\sigma \subseteq \tau} (-1)^{\dim(\tau)-\dim(\sigma)} \widetilde{d}_m(X/\sigma) \geq \sum_{\sigma \subseteq \tau} (-1)^{\dim(\tau)-\dim(\sigma)} \widetilde{d}_{m+1}(X/\sigma).$$

These conditions generalize to the relative setting the famous restrictions on the f-vector of a simplicial toric variety that come from Poincaré duality and Hard Lefschetz (see [12, Section 5.6]).

Remark 6.5. Suppose that $f \colon X \to Y$ is a proper, birational toric map. We may assume that $N_X = N_Y$ and f_N is the identity, hence Δ_X gives a fan refinement of Δ_Y. If for a cone $\tau \in \Delta_Y$ we define δ_τ by

$$(6.4) \qquad \delta_\tau = \sum_{\sigma \subseteq \tau} (-1)^{\dim(\tau)-\dim(\sigma)} d_0(X/\sigma),$$

then we want to give a "nonnegative expression" for δ_τ. For every cone $\tau \in \Delta_Y$, let $\iota(\tau)$ denote the number of rays in Δ that are contained in the relative interior of τ. If $\dim(\tau) \leq 3$, then we

have the following formulas:

(i) $\delta_0 = 1$ and $\delta_\tau = 0$ if $\dim(\tau) = 1$,

(ii) $\delta_\tau = \iota(\tau)$ if $\dim(\tau) = 2$,

(iii) $\delta_\tau = 2\iota(\tau)$ if $\dim(\tau) = 3$.

The assertions in (i) and (ii) follow easily from (6.4), hence we only prove (iii). In order to check this, it is convenient to consider a transversal section T of τ. This is a triangle such that Δ_X induces a triangulation Λ of T. Let us consider the following invariants:

(1) a_3 is the number of triangles in Λ,

(2) a_2 is the number of segments in Λ that are contained in the boundary of T,

(3) a_2' is the number of segments in Λ not contained in the boundary of T,

(4) a_1 is the number of points in Λ in the boundary of T,

(5) a_1' is the number of points in the interior of T (hence $a_1' = \iota(\sigma)$).

Note that we have the following relations between these invariants:

(R1) $(a_1 + a_1') - (a_2 + a_2') + a_3 = 1$ (by considering the Euler–Poincaré characteristic of T),

(R2) $a_1 = a_2$,

(R3) $3a_3 = a_2 + 2a_2'$ (by counting the segments in the boundaries of all triangles, and noting that a segment appears in two triangles if it is not contained in the boundary of T, and in one, otherwise).

By combining (R1) and (R3), we see that

$$3a_2 + 3a_2' - 3a_1 - 3a_1' + 3 = a_2 + 2a_2'.$$

Simplifying and using also R2), we obtain

(6.5) $$a_2' - a_1 - 3a_1' + 3 = 0.$$

On the other hand, it follows from (6.4) that $\delta_\tau = a_3 - a_2 + 2$. By using (R1) and (6.5), we obtain the desired conclusion

$$\delta_\tau = a_2' - (a_1 + a_1') + 3 = 3a_1' - a_1' = 2a_1' = 2\iota(\sigma).$$

It is worth noting that if $\dim(\tau) = 4$, then δ_τ is not a multiple of $\iota(\tau)$. Indeed, by considering the blow-up of $\mathbf{A}^4$ at the origin, we see that the only possibility would be $\delta_\tau = 3\iota(\tau)$. On the other hand, consider $f = g \circ h$, where $g\colon Z \to \mathbf{A}^4$ is the blow-up of an invariant line L, with exceptional divisor $E \simeq \mathbf{P}^2 \times L$ and h is the blow-up of Z along the subset $\mathbf{P}^2 \times \{0\} \subset E$. An easy computation shows that in this case $\iota(\tau) = 1$ but $\delta(\tau) = 4$.

7. Combinatorics of the toric Decomposition Theorem: the general case

Our goal in this final section is to determine the supports of an arbitrary toric fibration $f\colon X \to Y$ and to show that they are combinatorially determined. In this case there are two difficulties, compared with the setting in the previous section: on one hand, the poset structure of Δ_Y is more complicated; second, and more crucially, we need to take into account the

singularities of X and Y. These will come up through the local behavior of the intersection cohomology complexes. In order to deal with the latter issue we begin by introducing the following invariant of an arbitrary toric variety.

Let $\mathbb{Z}[T, T^{-1}]$ denote the ring of Laurent polynomials with integer coefficients. Given a toric variety Y and two cones $\tau \subseteq \sigma$ in Δ_Y, we define

$$R_{\tau,\sigma}(T) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Q}} \mathcal{H}^k(\mathrm{IC}_{V(\tau)})_{x_\sigma} T^k \in \mathbb{Z}[T, T^{-1}]$$

and

$$r_{\tau,\sigma} = R_{\tau,\sigma}(1) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Q}} \mathcal{H}^k(\mathrm{IC}_{V(\tau)})_{x_\sigma}.$$

Note that since the restriction of $\mathcal{H}^k(\mathrm{IC}_{V(\tau)})$ to each torus-orbit is constant by Lemma 5.6, we could have replaced in the above definition x_σ by any other point in $O(\sigma)$.

Remark 7.1. The function $R \colon \{(\tau, \sigma) \in \Delta_Y \times \Delta_Y : \tau \subseteq \sigma\} \to \mathbb{Z}[T, T^{-1}]$ only depends on the combinatorics of Δ_Y. Indeed, in order to see that $R_{\tau,\sigma}$ is combinatorially determined, we may replace Y by $V(\tau)$ and thus assume that $\tau = \{0\}$. In this case, the assertion is a consequence of [11, Theorems 1.1, 1.2] and [10, Theorem 6.2]. We also note that

$$R_{\tau,\sigma}(T) = T^{\dim(\tau)-n}$$

whenever $\dim(\sigma) - \dim(\tau) \leq 2$, where $n = \dim(Y)$. Indeed, in this case $V := V(\tau) \cap U_\sigma$ is a simplicial toric variety, hence $\mathrm{IC}_V = \mathbb{Q}_V[\dim(V)]$. In particular, since $R_{\tau,\tau}(T) = T^{\dim(\tau)-n}$ is invertible, it follows that the function R has an inverse

$$\widetilde{R} \colon \{(\tau, \sigma) \in \Delta_Y \times \Delta_Y : \tau \subseteq \sigma\} \to \mathbb{Z}[T, T^{-1}]$$

with respect to the convolution on the incidence algebra corresponding to the poset Δ_Y. We set $\widetilde{r}_{\tau,\sigma} = \widetilde{R}_{\tau,\sigma}(1)$.

The function $\widetilde{R}$ will feature in the description of the supports of a toric fibration.

Remark 7.2. It follows from [22, Proposition 8.1] that, up to signs and powers of T, the function $\widetilde{R}$ is just the function R associated with the dual poset. We are grateful to T. Braden for pointing this out to us.

Given a toric fibration $f \colon X \to Y$, we define the functions

$$P_f \colon \Delta_Y \to \mathbb{Z}[T, T^{-1}] \quad \text{and} \quad p_f \colon \Delta_Y \to \mathbb{Z}$$

by

$$P_{f,\sigma}(T) := \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Q}} H^k(f^{-1}(x_\sigma), \mathrm{IC}_X) T^k$$

and

$$p_{f,\sigma} := P_{f,\sigma}(1) = \dim_{\mathbb{Q}} H^*(f^{-1}(x_\sigma), \mathrm{IC}_X).$$

It is not a priori clear that $P_{f,\sigma}(T)$ and $p_{f,\sigma}$ are combinatorially determined, but this follows from Theorem 7.6 below. Finally, we define

$$S \colon \Delta_Y \to \mathbb{Z}[T, T^{-1}] \quad \text{as} \quad S_\tau(T) = \sum_{b \in \mathbb{Z}} s_{\tau,b} T^b,$$

where the $s_{\tau,b}$ are the multiplicities defined in Theorem 5.1. It follows from (i) in Theorem 5.1 that $S_\tau(T) = S_\tau(T^{-1})$. Using the invariants $P_{f,\sigma}$ and $\widetilde{R}_{\tau,\sigma}$, we can now describe the supports of any toric fibration.

Theorem 7.3. *Suppose that we are in the setting of Theorem 5.1. With the above definitions, for every $\tau \in \Delta_Y$, we have*

$$S_\tau(T) = \sum_{\sigma \subseteq \tau} \widetilde{R}_{\sigma,\tau}(T) P_{f,\sigma}(T).$$

Proof. We proceed as in the proof of Proposition 6.1. Consider the decomposition given by Theorem 5.1:

$$Rf_*(\mathrm{IC}_X) \simeq \bigoplus_{\tau \in \Delta_Y} \bigoplus_{b \in \mathbb{Z}} \mathrm{IC}_{V(\tau)}^{\oplus s_{\tau,b}}[-b].$$

Let $\sigma \in \Delta_Y$. By taking the stalk at x_σ and computing the i-th cohomology, we obtain

$$\dim_{\mathbb{Q}} H^i(f^{-1}(x_\sigma), \mathrm{IC}_X) = \sum_{\tau \subseteq \sigma} \sum_{b \in \mathbb{Z}} s_{\tau,b} \cdot \dim_{\mathbb{Q}} \mathcal{H}^{i-b}(\mathrm{IC}_{V(\tau)})_{x_\sigma},$$

which gives the equality

$$P_{f,\sigma}(T) = \sum_{\tau \subseteq \sigma} R_{\tau,\sigma}(T) S_\tau(T)$$

in $\mathbb{Z}[T, T^{-1}]$. Since $\widetilde{R}$ is the inverse of R with respect to convolution, we conclude

$$S_\tau(T) = \sum_{\sigma \subseteq \tau} \widetilde{R}_{\sigma,\tau}(T) \cdot P_{f,\sigma}(T).$$

This completes the proof.   □

Evaluating at $T = 1$ we find the following useful criterion for a stratum to be a support of the map f.

Corollary 7.4. *In the setting of Theorem 5.1, we have*

$$\delta_\tau = \sum_{\sigma \subseteq \tau} \widetilde{r}_{\sigma,\tau} \cdot p_{f,\sigma}.$$

Corollary 7.5. *If we are in the setting of Theorem 5.1 and if X is simplicial with $\dim(X) = d_X$, then for every $\tau \in \Delta_Y$, we have*

$$S_\tau(T) = \sum_{\sigma \subseteq \tau} \widetilde{R}_{\sigma,\tau}(T) \cdot \left(\sum_{m \geq 0} \left(\sum_{\ell \geq m} (-1)^{\ell-m} \binom{\ell}{m} d_\ell(X/\sigma) \right) T^{2m-d_X} \right)$$

and

$$\delta_\tau = \sum_{\sigma \subseteq \tau} \widetilde{r}_{\sigma,\tau} \cdot d_0(X/\sigma).$$

Proof. Since X is simplicial, we have $\mathrm{IC}_X = \mathbb{Q}_X[d_X]$. The first equality follows immediately from Theorem 7.3 and Corollary 4.7. The second equality is then obtained evaluating at $T = 1$:

$$p_{f,\sigma} = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{Q}} H^{i+d_X}(f^{-1}(x_\sigma), \mathbb{Q}) = \chi(f^{-1}(x_\sigma)) = d_0(X/\sigma).$$ □

By Remark 7.1, in order to show that the formulas for the invariants S_τ and δ_τ in Theorem 7.3 and Corollary 7.4 only depend on combinatorics, it is enough to show that the invariants $P_{f,\sigma}$ only depend on combinatorics. This is implied by the following theorem. When Y is a point and X is projective, this is a consequence of the results in [11].

Theorem 7.6. *If $f: X \to Y$ is a toric fibration, then for every $\sigma \in \Delta_Y$, we have*

$$(7.1) \qquad P_{f,\sigma}(T) = \sum_\tau R_{0,\tau}(T)(T^2 - 1)^{\operatorname{codim}(\tau) - \operatorname{codim}(\sigma)} \quad \text{and} \quad p_{f,\sigma} = \sum_\tau r_{0,\tau},$$

where in the first formula the sum is over all the cones τ in Δ_X with $f_*(\tau) = \sigma$, while in the second formula the τ are only those which, in addition, satisfy $\operatorname{codim}(\tau) = \operatorname{codim}(\sigma)$.

Proof. The second statement follows from the first by evaluating it at $T = 1$, so it is enough to prove the first statement.

First we prove that $H^i(f^{-1}(x_\sigma), \operatorname{IC}_X)$ is pure, so that it is enough to determine its Hodge–Deligne polynomial. After replacing Y by U_σ, we may assume that $Y = U_\sigma$. Moreover, it is easy to see using Lemma 2.6 that we may assume that x_σ is a fixed point. In this case, it follows from Lemma 4.2 (see also Remark 4.4) that

$$H^i(f^{-1}(x_\sigma), \operatorname{IC}_X) = H^k(X, \operatorname{IC}_X),$$

therefore, as discussed in Remark 3.8, $H^i(f^{-1}(x_\sigma), \operatorname{IC}_X)$ is pure.

We proceed as in the proof of Corollary 4.7. Set $n = \dim X$. By Lemma 5.6, the restriction of the complex IC_X to every torus orbit $O(\tau)$ in $f^{-1}(y)$ is a complex with *constant cohomology sheaves* $\mathcal{H}^k(\operatorname{IC}_X)_{x_\tau}$, underlying a pure Hodge–Tate structure of weight $k + n$.

Set $t = \dim O(\tau)$. Since

$$H_c^p(O(\tau)) \cong \mathbb{Q}(p - t)^{\oplus \binom{t}{p}},$$

the compact cohomology group $H_c^p(O(\tau), \mathcal{H}^q(\operatorname{IC}_X)_{x_\tau})$ has a Hodge structure of Hodge–Tate type and weight $2(p - t) + q + n$. As is well known, see [8, p. 571] or [3, Example 5.2 (1)], the differentials in the hypercohomology spectral sequence

$$E_2^{p\,q} = H_c^p(O(\tau), \mathcal{H}^q(\operatorname{IC}_X)_{x_\tau}) \Rightarrow H_c^{p+q}(O(\tau), \operatorname{IC}_X)$$

are compatible with the Hodge structure , and they are therefore forced to vanish. It follows that the Hodge–Deligne polynomial of $H_c^{p+q}(O(\tau), \operatorname{IC}_X)$ is $R_{0,\tau}(T)(uv - 1)^t$. Adding over all torus orbits contained in $f^{-1}(x_\sigma)$ and setting $u = v = T$, we obtain the statement. $\quad \square$

Remark 7.7. Due to the combinatorial nature of the definition of the local h-polynomial in [15], the proofs of the analogues of Theorems 7.3 and 7.6 in that setting are more elementary. One can then use the results of this section to write our invariants $s_{\tau,b}$ as coefficients of local h-polynomials.

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